



Quantifying effects from extreme events with applications to financial crises

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Abstract

This manuscript addresses the quantification of effects from an extreme outcome in an explanatory variable on a dependent variable. The effect is approximated with the so-called asymptotic elasticity of a conditional quantile function, linking the dependent and explanatory variable. A closed form expression for this asymptotic elasticity is presented which is independent of the exact relation between explanatory and dependent variable. By interpreting the asymptotic elasticity as a spill-over measure for tail-risk, we detect statistically significant effects from Lehman Brothers to other financial institutions during the subprime mortgage crisis before Lehman Brothers was obviously in distress. Likewise, the effect from a credit default in case of Greece on the solvency of countries within the Euro-area is briefly studied.

Keywords: Conditional quantile, Copula, Asymptotic elasticity, Spill-over, Tail-risk.

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1. Introduction

The subprime mortgage crisis of 2007-2008 has revealed the need to appropriately model and measure interconnectedness between financial institutions. Recent research on systemic risk illustrates that increased connectivity among financial institutions causes additional risk through a complex and time-varying network of relationships, see, e.g., [Billio, Getmansky, Lo, and Pelizzon \(2012\)](#); [Acemoglu, Ozdaglar, and Tahbaz-Salehi \(2015\)](#). As a result, risk managers have a strong interest in the risk transmitted to their institution from other institutions, as it happened, for example, during the global financial crisis. This additional risk exposure is typically characterized by high uncertainty and is difficult to control, since it originates in the failure or distress of related financial institutions. Moreover, policy makers and regulators aim at identifying risk factors that will enable them to react to situations of market stress in a suitable manner. Thus, following recent crises in financial markets many concepts for quantifying the transmission of risks and measuring systemic risk have been proposed in the literature, see, e.g., [Brunnermeier and Oehmke \(2013\)](#) for a summary of the empirical literature, [Bisias, Flood, Lo, and Valavanis \(2012\)](#) for a survey on quantitative approaches to the measurement of systemic risk, and [Benoit, Colliard, Hurlin, and Pérignon \(2017\)](#) for a comprehensive structural review of the extensive literature.

Particular interest lies in addressing questions such as: “How is a financial institution affected by a bad outcome or extreme event related to another financial institution?”. The likelihood and magnitude of such an extreme event can be quantified by knowing the tail of the distribution of the corresponding risk factor. Likewise, the stability or fragility of an affected institution can be assessed by examining the tail of its own risk factor distribution. Thus, a conditional quantile linking the tails of both distributions is a natural candidate to describe how the tail-risk of one risk factor depends on the tail-risk of another risk factor.

Conditional quantiles have become a standard tool for modeling the tail-risk of a dependent variable through the tail-risk of an explanatory variable, see, e.g., the seminal paper of [Adrian and Brunnermeier \(2016\)](#) as well as the references therein. The authors propose the Δ CoVaR measure which is the difference between the conditional quantile given that the explanatory

variable is at its Value at Risk (VaR) and the conditional quantile given that the explanatory variable is at its median. The literature, yet, offers a wide range of approaches for measuring the corresponding spill-over from tail-risks. For example, a number of studies utilize the marginal effect of a function linking dependent and explanatory variables as a spill-over measure for tail-risk, see, e.g., [Hautsch, Schaumburg, and Schienle \(2015\)](#); [Härdle, Wang, and Yu \(2016\)](#); [Betz, Hautsch, Peltonen, and Schienle \(2016\)](#).

Extending the ideas of the above-mentioned studies, we propose the use of the *normalized* derivative of a conditional quantile with respect to its explanatory variable as a spill-over measure, that is a well-known concept in economics referred to as elasticity. Compared to elasticities, plain marginal effects do not allow categorizing a spill-over effect as weak or strong which follows from the fact that marginal effects have no reference value for comparison. An elasticity, however, measures the magnitude of a function's responsiveness and has a simple interpretation due to its natural reference value, e.g., see [Sydsæter and Hammond \(1995\)](#).

To quantify effects from an extreme event, e.g., a drastic drop in the share price of a company, a slightly modified version of an elasticity is needed though. An ordinary elasticity of a conditional quantile with the explanatory variable being at its limit defines a so-called asymptotic elasticity, which is by definition properly designed to quantify effects from a worst-case scenario. Thus, our spill-over measure for tail-risk is defined by an asymptotic elasticity of a conditional quantile. Note that the asymptotic elasticity is particularly suitable for addressing hypothetical questions of the type "What if ...?". In the considered economic context, this translates into "What is the effect on the tail-risk of financial institution A, if the share price of financial institution B drops substantially?".

In the developed framework, we provide a simple representation for the asymptotic elasticity by imposing two local requirements on the conditional quantile: (i) The distributions of the dependent and explanatory variable exhibit tail-monotone densities. Distributions of this class are endowed with a tail-exponent controlling the decay of the tail. (ii) The underlying conditional quantile is required to be monotonically increasing with bounded derivatives. If these conditions are met in the tail-area under investigation, a simple formula for the asymptotic elasticity is obtained as the ratio of the tail-exponents of the dependent and

explanatory variable. This representation is independent of the specific functional form of the conditional quantile except for the imposed requirements.

To get a more intuitive understanding of the asymptotic elasticity, consider the returns of two financial institutions A and B with different levels of risk. Let the returns of financial institution A be the dependent variable, and the returns of the institution B be the explanatory variable. Given that both companies belong to the same financial sector, the corresponding conditional quantile can be reasonably assumed to have a positive slope. Moreover, if the distribution of the returns of institution A is relatively heavy-tailed compared to that of institution B, the slope of the conditional quantile appears steep. This is simply because there is more “dispersion” in the dependent than in the explanatory set of stock returns. In such a setting, the impact of an extreme event in the returns of institution B can be quantified via the ratio of measures for the stock returns’ marginal behaviors in extremes in terms of the tail-exponents. This relation is the asymptotic elasticity.

To appreciate possible benefits of the simple representation for the asymptotic elasticity, note that the ordinary elasticity can in fact be estimated by replacing the underlying conditional quantile through a corresponding estimate. Confidence intervals of estimated conditional quantiles, however, become usually wider as the level of the conditional quantile becomes more extreme. This unfavorable property is obviously transferred to the confidence intervals of the corresponding elasticity. In contrast, the simple formula for the asymptotic elasticity results in a convenient situation, where a point instead of a curve is to be estimated. Even though the approximated distribution for the estimated asymptotic elasticity is based on a parametric assumption on the distributions of the dependent and the explanatory variable, the conditional quantile itself does not need to be specified.

In an empirical study, we apply the developed modeling approach to measure the tail-risk transferred to other financial institutions as consequence of the collapse of Lehman Brothers. Overall, we conclude that the bankruptcy of Lehman Brothers was an outstanding big event, since originally it seemed like Lehman Brothers had mastered the subprime crisis relatively well compared to other financial institutions. One of our main findings reveals that Lehman Brothers became systemically relevant for the list of other financial institutions only in July

2008. Before this date the shock emitted in case of a collapse of Lehman Brothers was not significantly stronger than the shock Lehman Brothers had received in case of a collapse of other institutions. We also illustrate the differences between the newly developed measure and traditional measures of spill-over effects from tail-risks such as CoVaR. To do this, in our empirical study, we compute a version of the Δ CoVaR measure based on the same time period and estimated models.

In a second case study, our approach is applied to the European sovereign debt crisis. [Lucas, Schwaab, and Zhang \(2014\)](#) estimate probabilities of a credit default for countries of the Euro-area conditional on the extreme event that Greece defaults. The authors find that the conditional default probability due to a credit event in Greece decreases while the default probability for Greece increases. As the asymptotic elasticity is derived from a conditional quantile instead of a conditional probability, our results are in some way complementary to those of [Lucas *et al.* \(2014\)](#). For the period of decreased conditional default probability, we confirm the findings of [Lucas *et al.* \(2014\)](#) by revealing a reduction in the spill-over of tail-risk from Greece to the other countries, based on underlying differenced prices of credit default swaps on sovereign debt.

The remainder of the paper is organized as follows. [Section 2](#) introduces the basic notation and briefly reviews some of the existing approaches that have been proposed in the literature to measure spill-over effects for tail-risk relying on a conditional quantile function. [Section 3](#) is devoted to the concept of asymptotic elasticity and provides the underlying ideas as well as some theoretical results to illustrate the main intuition for applications of the developed measure. Note that most of the technical details such as discussions of assumptions as well as the proofs of the results of [Section 3](#) have been moved to [Appendix A](#). [Section 4](#) presents empirical results from applying our measure to returns of financial institutions during the subprime crisis, while [Section 5](#) examines the European sovereign debt crisis. [Section 6](#) concludes.

2. CoVaR and related concepts

Let the random variables $X_1, X_2 \in \mathbb{R}$ refer to two risk factors of financial institutions, where “good events” like profits are on the negative part and “bad events” such as losses on the positive part of the real line. Denote by $F(x_1, x_2) = \text{P}(X_1 \leq x_1, X_2 \leq x_2)$ the joint cumulative distribution function (cdf) of X_1 and X_2 . The cdf $F(x_1, x_2)$ is assumed to be continuously differentiable and strictly increasing in both arguments. The corresponding conditional cdf is defined by

$$F_{X_2|X_1=x_1}(x_2) = \text{P}(X_2 \leq x_2 | X_1 = x_1). \quad (1)$$

As $F_{X_2|X_1=x_1}(x_2)$ is strictly increasing in x_2 , the conditional quantile function is given by

$$Q_{X_2|X_1=x_1}(\alpha) = F_{X_2|X_1=x_1}^{-1}(\alpha) \quad \text{for fixed } \alpha \in (0, 1). \quad (2)$$

We denote by $F_j(x_j)$ and $Q_j(u_j) = F_j^{-1}(u_j)$, $u_j \in (0, 1)$, the marginal cdf of $F(x_1, x_2)$ and quantile function, $j = 1, 2$.¹

To put these formulas in an economic context, in the following we relate them to the CoVaR approach and similar concepts, where conditional quantiles are used as risk measures. [Adrian and Brunnermeier \(2016\)](#) measure the contribution from risk factor X_1 to risk factor X_2 via

$$\Delta \text{CoVaR} = Q_{X_2|X_1=Q_1(\alpha)}(\alpha) - Q_{X_2|X_1=Q_1(0.5)}(\alpha)$$

for α being close to one. For instance, if the risk factors refer to negative log-returns of a financial institution, ΔCoVaR denotes the effect on the VaR for X_2 from a change in the riskiness of X_1 from the median to an extreme level of risk.

Different to the CoVaR approach, the concepts of [Hautsch *et al.* \(2015\)](#) as well as [Härdle *et al.* \(2016\)](#) combine marginal effects derived from the conditional quantile with arguments from network theory to determine spill-over effects and systemic risk measures. For instance,

¹While the major parts of this manuscript concentrate on the case of two random variables, we offer a short discussion on the multivariate setting in Section 3.4 which is formally inspected in Appendix A.

Härdle *et al.* (2016) construct a $(d \times d)$ adjacency matrix consisting of marginal effects from d conditional quantile curves representing a systemic risk network. By importing the interpretation of Diebold and Yilmaz (2014), so called systemic risk emitters and receivers are then derived from this adjacency matrix.

As we illustrate in Appendix A, the partial derivative of a conditional quantile with respect to (wrt) the explanatory variable can be disentangled into three major components: one contribution stemming from the distribution of the dependent variable, one from the explanatory variable and one from their dependence structure. The conceptual advantage of the approaches of Hautsch *et al.* (2015) and Härdle *et al.* (2016) over Δ CoVaR is that measuring the spill-over of tail-risks via marginal effects takes all three components into account. In contrast, the Δ CoVaR is entirely unrelated to the distribution of the explanatory variable.

Remark 1. *A critical conceptual discussion of the CoVaR approach is provided by Mainik and Schaanning (2014). Statistical shortcomings of the CoVaR approach such as the omitted variables bias and the linear specification of the conditional quantile are addressed in Hautsch et al. (2015) and Härdle et al. (2016) respectively. The link between Δ CoVaR and multivariate GARCH models is analyzed in Girardi and Ergün (2013).*

3. Quantifying effects from extreme events

As illustrated above, marginal effects can be considered as a rewarding concept as it comes to the quantification of the spill-over effects for tail-risk. However, conclusions about the strength of spill-over effects are difficult to be drawn from these marginal effects. Even if a marginal effect is statistically found to be different from zero, it is not clear whether the effect is weak or strong. In this section we introduce a new measure for spill-over effects that overcomes some of these drawbacks.

3.1. Asymptotic elasticity

Let us define a spill-over measure by normalizing the derivative of the conditional quantile

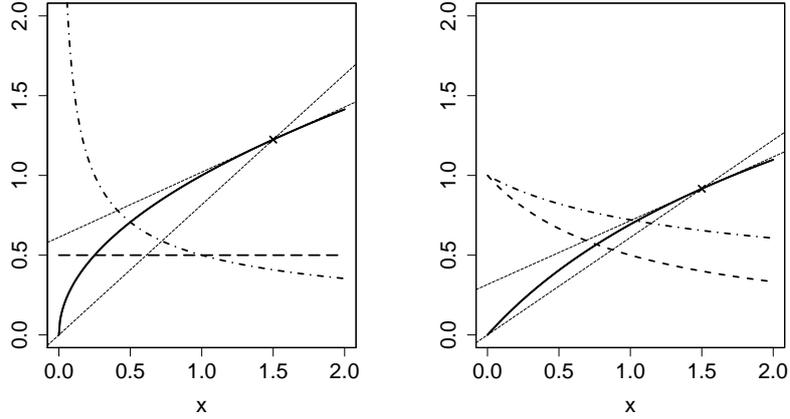


Figure 3.1: Illustration of elasticity functions. The left panel illustrates the mapping $f(x) = \sqrt{x}$ (solid), the tangent at $x_0 = 3/2$ as well as a straight line from the origin to the point $(x_0, f(x_0))$ (both thin-dashed), the derivative $f'(x)$ (dashed-dotted) and the elasticity function $\mathcal{E}(x) = xf'(x)/f(x)$ (dashed). The right panel shows corresponding curves for $f(x) = \log(1+x)$.

$\frac{\partial}{\partial x_1} Q_{X_2|X_1=x_1}(\alpha)$ through division by $Q_{X_2|X_1=x_1}(\alpha)/x_1$, i.e.,

$$\mathcal{E}(x_1) = \frac{x_1}{Q_{X_2|X_1=x_1}(\alpha)} \frac{\partial}{\partial x_1} Q_{X_2|X_1=x_1}(\alpha) \quad \text{for fixed } \alpha \in (0, 1), \quad (3)$$

where the normalizing factor is the slope of the line from the origin to the point $(x_1, Q_{X_2|X_1=x_1}(\alpha))$. Functions of form (3) are commonly considered as elasticity, see [Sydsæter and Hammond \(1995\)](#).

To get a better understanding of the elasticity function, consider the mapping $f(x) = \sqrt{x}$ defined on the positive real line and depicted as solid curve in the left panel of Figure 3.1. The thin-dashed lines refer to the tangent at $x_0 = 3/2$ as well as the extended chord connecting the origin of the coordinate plane and the point $(x_0, f(x_0))$. Intuitively, the derivative $f'(x)$ - depicted as dashed-dotted line - can be determined by the slope of the tangent for any point x . Likewise, the corresponding elasticity function $\mathcal{E}(x) = xf'(x)/f(x)$ - depicted as dashed (horizontal) line - can be determined by the ratio of the slopes of the tangent and the extended chord line for any point x . While $f'(x)$ measures the *absolute* change in $f(x)$ due to a (small) *absolute* change in x , the elasticity $\mathcal{E}(x)$ measures the *relative* change in $f(x)$ due to a (small) *relative* change in x . This allows the following interpretation: (i) If $|\mathcal{E}(x)| < 1$, the function

$f(x)$ is called inelastic or robust wrt x ; (ii) if $|\mathcal{E}(x)| > 1$, the function $f(x)$ is called elastic or responsive wrt x ; (iii) if $|\mathcal{E}(x)| = 1$, the function $f(x)$ is called proportionally elastic wrt x . Note that for both examples in Figure 3.1, $f(x)$ is inelastic wrt x . However, while the elasticity function $\mathcal{E}(x)$ is constant for $f(x) = \sqrt{x}$, it is decreasing in x for $f(x) = \log(1+x)$. To interpret the elasticity of a conditional quantile $\mathcal{E}(x_1)$ from (3) in terms of the considered economic context, keep in mind that X_1 and X_2 represent risk factors such as negative log-returns of financial institutions. To study the responsiveness of the tail-risk of X_2 wrt to the tail-risk of X_1 , the level $\alpha \in (0, 1)$ in the definition of the elasticity is typically chosen close to one. Moreover, let x_0 be a value related to the tail-risk of X_1 , e.g., the VaR of X_1 . Using the interpretation for marginal effects of Härdle *et al.* (2016), the elasticity $\mathcal{E}(x_0)$ can be interpreted as a measure for the tail-risk emitted from financial institution with risk factor X_1 to the tail-risk of institution with risk factor X_2 .

Analogously to the example above, we obtain the following interpretation for the considered application: (i) If the spill-over measure for tail-risk tends to zero $|\mathcal{E}(x_0)| \ll 1$, the tail-risk of financial institution with risk factor X_2 is said to be robust wrt the tail-risk of X_1 . Equivalently, the institution with risk factor X_1 cannot be categorized as a risk emitter. (ii) If $|\mathcal{E}(x_0)| \gg 1$, the tail-risk of institution with risk factor X_2 is said to be responsive or fragile wrt the tail-risk of risk factor X_1 . Equivalently, X_1 can be categorized as a risk emitter wrt X_2 . (iii) If $|\mathcal{E}(x_0)| \approx 1$, the tail risk of X_2 is proportionally elastic wrt the tail-risk of X_1 .

According to (3), the presented elasticity $\mathcal{E}(x_1)$ can be derived from the conditional quantile for fixed $\alpha \in (0, 1)$ being usually close to one. However, interesting cases such as extreme conditioning events, where $x_1 \rightarrow \infty$, are difficult to study. For instance, if X_1 refers to a negative log-return, an extreme event in X_1 reflects a drastic drop in the stock price of a company to zero. An elasticity related to such an extreme conditioning event is given by

$$\mathcal{E} = \lim_{x_1 \rightarrow \infty} \mathcal{E}(x_1). \quad (4)$$

An expression of form (4) is called asymptotic elasticity and has been introduced in a different context by Kramkov and Schachermayer (1999). Given that in financial applications we

would typically be interested in the spill-over effects of extreme losses, for convenience, in the following we concentrate on the “right tail” ($x_1 \rightarrow \infty$) of the loss distribution and neglect the “left tail” ($x_1 \rightarrow -\infty$).

Note, however, that for empirical applications, (4) has to be interpreted as an approximation of the spill-over from an extreme event. For instance, even in the event of default, a stock price typically does not drop to zero, but rather to a relatively small value. Hence, the spill-over effect for tail-risk is eventually quantified too large in certain situations. We would argue though, that for the purpose of risk management, an overestimation of the transferred risk between financial institutions can be seen as less harmful than its underestimation.

Remark 2. *The proposed approach based on the asymptotic elasticity and the line of arguments we present below are rather unrelated to methods dealing with extremal dependence such as conditional extreme value theory and extremal quantile regression, see, e.g., [Heffernan and Tawn \(2004\)](#) and [Chernozhukov \(2005\)](#), respectively.*

3.2. Properties of the asymptotic elasticity

To characterize properties of the asymptotic elasticity \mathcal{E} , we import the concepts of a tail-monotone density function with associated tail-exponent and conditional tail-(in)dependence discussed in Appendix A.

Definition 1 (Tail-monotonicity, see [Parzen \(1979\)](#)). *Let $F(x)$ and $f(x) = F'(x)$ be the cdf and density function of a random variable $X \in \mathbb{R}$. The density $f(x)$ is called tail-monotone, if it is non-decreasing on an interval to the right of $a = \sup\{x : F(x) = 0\}$ and non-increasing on an interval to the left of $b = \inf\{x : F(x) = 1\}$, with $-\infty \leq a \leq b \leq \infty$; and if $f(x) > 0$ on $x \in (a, b)$ and $\sup_{x \in (a, b)} F(x)\{1 - F(x)\}|f'(x)|/f(x)^2 \leq \gamma$, where the tail-exponent is $\gamma = \lim_{x \rightarrow \infty} \log f(x)/\log\{1 - F(x)\} > 0$.*

In the considered context, the density of a random variable is intuitively called tail-monotone if it is positive and strictly decreasing on an interval (x_0, ∞) for some x_0 . The associated tail-exponent $\gamma \geq 1$ characterizes (i) exponential- and (ii) long tails for (i) $\gamma = 1$ and (ii)

$\gamma > 1$. Numerous probability laws have tail-monotone densities such as the Gaussian ($\gamma = 1$), Cauchy ($\gamma = 2$), Pareto ($\gamma = 1 + 1/\beta$) and Student's- t_ν ($\gamma = 1 + 1/\nu$) distributions, where $\beta \in (0, \infty)$ and $\nu \in [1, \infty)$ denote the parameters of the respective distribution. Density functions which oscillate in the tails are not tail-monotone, see Parzen (1979, p. 117) for an example. The remainder of the manuscript is restricted to tail-monotone densities whose tail-exponents satisfy $\gamma \geq 1$. The tail-exponent of a distribution with a short-tailed and tail-monotone density satisfies $0 < \gamma < 1$, but the corresponding random variable might not be suitable for our modeling purpose, e.g., the distribution has eventually bounded support, see Parzen (1979, p. 115).

Remark 3. *The considered tail-exponent γ is different from the classical tail-index, e.g., see Embrechts, Klüppelberg, and Mikosch (1997). Relations between both are studied in Holan and McElroy (2010, Section 2) who observe that densities with exponentially decaying tails are not covered by the definition of the classical tail-index but by that of the tail-exponent γ .*

Let the notation $z(x) \sim y(x)$, $x \rightarrow a$, mean $\lim_{x \rightarrow a} z(x)/y(x) = K$, with K being a positive finite constant.

Definition 2 (Conditional tail-(in)dependence). *The random variables $X_2 \in \mathbb{R}$ and $X_1 \in \mathbb{R}$ are called conditionally independent in the right tail, if $Q_{X_2|X_1=x_1}(\alpha) \sim g(\alpha)$, $x_1 \rightarrow \infty$, where $g(\alpha)$ is a constant depending on $\alpha \in (0, 1)$. Likewise, X_2 and X_1 are called conditionally tail-dependent if $Q_{X_2|X_1=x_1}(\alpha) \rightarrow \infty$ as $x_1 \rightarrow \infty$ such that the derivative of $Q_{X_2|X_1=x_1}(\alpha)$ wrt x_1 is strictly positive and bounded on an interval (x_0, ∞) for some x_0 .*

The intuition of this notion of conditional tail-(in)dependence is as follows: The risk factors are basically called conditionally tail-independent, if the conditional quantile $Q_{X_2|X_1=x_1}(\alpha)$ becomes flat for large values in the explanatory variable. Conversely, the risk factors are called conditionally tail-dependent, if the conditional quantile steadily increases in the explanatory variable, but not too fast. It should be noticed that the technical requirement that the derivative of $Q_{X_2|X_1=x_1}(\alpha)$ wrt x_1 is bounded on the specified interval is required for the identification of effects arising from extreme events. If this derivative is not bounded on

the considered interval, the conditional quantile $Q_{X_2|X_1=x_1}(\alpha)$ approaches infinity for non-extreme outcomes in X_1 . In this situation however, the effect of an extreme outcome in X_1 cannot be uniquely identified.

Proposition 1. *Let $F(x_1, x_2)$ be continuously differentiable and strictly increasing in x_1 and x_2 . Let X_2 and X_1 have tail-monotone densities with tail-exponents γ_2 and γ_1 .*

(a) *If X_2 and X_1 are conditionally tail-independent and $\gamma_2 \geq 1$, $\gamma_1 \geq 1$, then $\mathcal{E} = 0$.*

(b) *If X_2 and X_1 are conditionally tail-dependent and $\gamma_2 > 1$, $\gamma_1 = 1$, then $\mathcal{E} = \infty$.*

(c) *If X_2 and X_1 are conditionally tail-dependent and $\gamma_2 \geq 1$, $\gamma_1 > 1$, then $\mathcal{E} = \frac{\gamma_2 - 1}{\gamma_1 - 1}$.*

As expected and summarized in part (a) of Proposition 1, if X_2 and X_1 are conditionally tail-independent, there is no effect from an extreme event in the explanatory variable on the dependent variable. Even though this stresses the importance of conditional tail-dependence for analyzing the relation between tails of univariate distributions, the more interesting results are documented in part (b) and (c) of Proposition 1.

On an abstract level, part (b) of Proposition 1 implies that an extreme event in the explanatory variable entails an extreme event in the dependent variable. Let us now consider this relationship in an economic context: Let X_1 refer to the negative log-return of a financial institution which has never been in distress, so that the right tail of the probability law of X_1 can be assumed to decay exponentially, i.e., $\gamma_1 = 1$. Let X_2 refer to the negative log-return of a financial institution which is occasionally in distress, such that the right tail of the probability law of X_2 satisfies $\gamma_2 > 1$. Now, part (b) of Proposition 1 can be interpreted as follows: If the “low-risk” financial institution is in distress, the tail-risk of the “high-risk” financial institution reacts sensitive wrt the distress of the low-risk institution.

Conditional quantiles per definition depend on the joint distribution between dependent and explanatory variables which can be uniquely determined by marginal distributions and the so called copula function. Figures 3.2 and 3.3 show in total four pairs of copula-based conditional quantiles and corresponding elasticities. Even though the construction of conditional quantiles from copulas is formally introduced in Appendix A, copula-based quantiles are not required

to be known for the interpretation of Figures 3.2 and 3.3. This follows simply from the fact that they are neither required to be known for the understanding of Proposition 1. Thus, we content ourselves with knowing that the four presented conditional quantiles link the risk factors X_1 and X_2 in fairly different ways. However, the tail-exponents are required to be known for the interpretation of Figures 3.2 and 3.3.

The situations explained above referring to part (a) and (b) of Proposition 1 are illustrated in the left and right panels of Figure 3.2 respectively. Since a coordinate plane cannot approach infinity, the abscissae of the elasticity graphics in Figures 3.2 and 3.3 refer to the respective cdf of X_1 , which offers a simple way to show how the elasticity function from (3) approaches the corresponding limit from Proposition 1.

The left panels of Figure 3.2 rely on conditionally tail-independent risk factors X_1 and X_2 . Risk factor X_1 is Gaussian distributed with $\gamma_1 = 1$ and X_2 is Cauchy distributed with $\gamma_2 = 2$, so that the conditions of Proposition 1(a) are fulfilled. Even though the conditional quantile curves in the upper-left panel of Figure 3.2 suggest some dependence between X_1 and X_2 , the elasticity functions in the lower-left panel of Figure 3.2 illustrate that there is no effect to the corresponding α -quantile of X_2 in case of an extreme event in X_1 . The right panels of Figure 3.2 are based on conditionally tail-dependent risk factors X_1 and X_2 being still Gaussian and Cauchy distributed. Since this setup satisfies the requirements of Proposition 1(b), the corresponding elasticity functions become infinite as $x_1 \rightarrow \infty$.

The most useful result of Proposition 1 is part (c) stating the exact value of the asymptotic elasticity in case that the probability laws of X_1 and X_2 are long-tailed. The intuition formed for part (b) applies here as well: A risk factor X_2 , whose distribution is relatively long-tailed compared to that of X_1 , reacts responsive to the tail-risk of X_1 . Conversely, a risk factor X_1 , whose distribution is relatively long-tailed compared to that of X_2 , transfers merely small proportions of tail-risk.

For instance, let the risk factors X_2 and X_1 be Student's- t_ν distributed with parameters $\nu_2 \geq 1$ and $\nu_1 \geq 1$ respectively and suppose they are linked with a conditional quantile having a positive slope. If we agree on measuring the effect from an extreme event in the explanatory

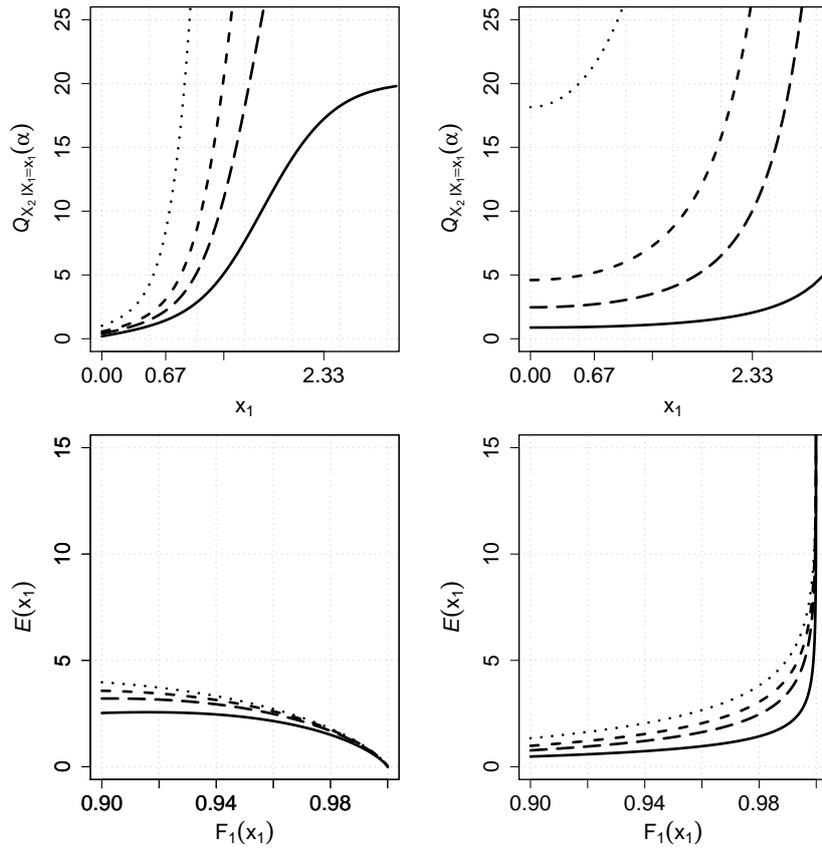


Figure 3.2: Conditional quantiles and elasticities from Equations 2 and 3. The Gaussian (X_1) and Cauchy (X_2) distributed margins are coupled with a Frank copula in the left panels and with a Student's- t copula in the right panels. The lines refer to the levels $\alpha = 0.75$ (solid), $\alpha = 0.90$ (long-dashed), $\alpha = 0.95$ (dashed) and $\alpha = 0.99$ (dotted).

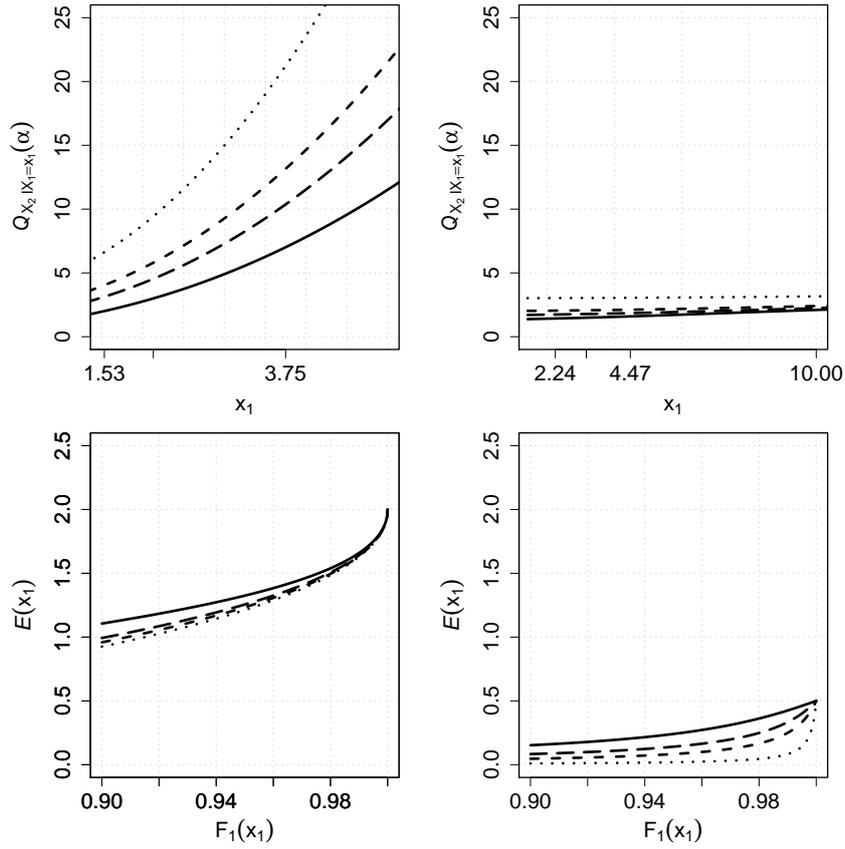


Figure 3.3: Conditional quantiles and elasticities from Equations 2 and 3. The left panels are based on Student's- t_ν distributed margins with $\nu_1 = 4$ and $\nu_2 = 2$ coupled with a survival Clayton copula. The right panels rely on Pareto distributed margins with $\beta_1 = 2$ and $\beta_2 = 4$ coupled with a Cauchy-type copula. The lines refer to the levels $\alpha = 0.75$ (solid), $\alpha = 0.9$ (long-dashed), $\alpha = 0.95$ (dashed) and $\alpha = 0.99$ (dotted).

variable via the asymptotic elasticity, it can be approximated by ν_1/ν_2 . Note that the smaller the parameter of the Student's- t_ν distribution the longer are its tails. Hence, a large (small) value of the fraction ν_1/ν_2 refers to a situation where pairs of observations stretch rather vertically (horizontally) in the coordinate plane.

Figure 3.3 shows the asymptotic behavior of the elasticity function related to part (c) of Proposition 1. In the left panels, the risk factors X_1 and X_2 are conditionally tail-dependent and follow a Student's- t_ν distribution with $\nu_1 = 4$ and $\nu_2 = 2$ degrees of freedom yielding $\gamma_1 = 5/4$ and $\gamma_2 = 6/4$ respectively. Thus, Proposition 1(c) results in $\mathcal{E} = 2$. As shown in the graphic, $\mathcal{E} = 2$ is exactly the limit of $\mathcal{E}(x_1)$ as $x_1 \rightarrow \infty$. The conditionally tail-dependent risk factors X_1 and X_2 underlying the right panels of Figure 3.3 are Pareto distributed with parameters $\beta_1 = 2$ and $\beta_2 = 4$ which imply the tail-exponents $\gamma_1 = 6/4$ and $\gamma_2 = 5/4$ respectively. Thus, Proposition 1(c) leads to the result $\mathcal{E} = 1/2$. As depicted in the lower right panel of Figure 3.3, $\mathcal{E} = 1/2$ is the limit of the ordinary elasticity $\mathcal{E}(x_1)$ as $x_1 \rightarrow \infty$.

These graphics illustrate remarkably that the asymptotic elasticity does not depend on the function linking the risk factors, apart from the requirements on the conditional quantile which are discussed in Appendix A for this example. However, the imposed requirements on the conditional quantile might depend on the value $\alpha \in (0, 1)$ and other parameters of the function linking the risk factors, see Appendix A.

Overall, we conclude that in order to quantify the effect arising from an extreme event, the relation of the tails of the risk factors' distributions is more important than the specific functional dependence between these risk factors.

3.3. Weights

So far, the risk factors X_2 and X_1 have not been weighted, e.g., according to market capitalization of corresponding financial institutions. Some studies highlighted above try to take effects arising from different importance of financial institutions via weights into account, see, e.g., Adrian and Brunnermeier (2016); Härdle *et al.* (2016). We do not include weights in our analysis for a simple reason: Elasticities derived from conditional quantiles are invariant wrt the scaling of the underlying risk factors. An alternative representation of the asymp-

otic elasticity immediately reveals that the consideration of weights does not influence the asymptotic elasticity, see Formula 11 in Appendix A. Note also that López-Espinosa, Moreno, Rubia, and Valderrama (2012) find empirical evidence that the size of international banks is of weak relevance for systemic risk.

3.4. Several explanatory variables

Up to now, the analysis has concentrated on the case of one dependent and one explanatory variable. Extending the framework to a multivariate setting including d risk factors X_1, \dots, X_d raises the question how the asymptotic elasticity involving two risk factors is affected by ignoring the remaining risk factors.

Let $\mathcal{E}(x_\ell)$ be the elasticity from (3), measuring the effect from X_ℓ to the α -quantile of X_k , where other risk factors exist but are excluded from the analysis. In a multivariate setting that includes all risk factors in the analysis, the elasticity measuring effects from X_ℓ to the α -quantile of X_k is denoted by

$$\mathcal{E}_{k\ell}(x_{-k}) = \frac{x_\ell}{Q_{X_k|X_{-k}=x_{-k}}(\alpha)} \frac{\partial}{\partial x_\ell} Q_{X_k|X_{-k}=x_{-k}}(\alpha), \quad \alpha \in (0, 1), \quad (5)$$

where the conditional quantile with several explanatory variables $Q_{X_k|X_{-k}=x_{-k}}(\alpha)$ can be analogously defined as in the bivariate case, see Appendix A, and the variable x_k is not included in $x_{-k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d)^\top$.

The asymptotic elasticity $\mathcal{E} = \lim_{x_\ell \rightarrow \infty} \mathcal{E}(x_\ell)$ relying on two risk factors X_k and X_ℓ measures precisely the same effect as $\mathcal{E}_{k\ell}(x_{-k})$, if *all* included explanatory variables are subject to an extreme event. In other words, the asymptotic elasticity \mathcal{E} is equivalent to $\mathcal{E}_{k\ell}(x_{-k})$ given that all components of x_{-k} in (5) converge simultaneously to infinity. From this perspective, the bilateral asymptotic elasticity \mathcal{E} can always be interpreted as the effect from the ℓ -th explanatory variable in a worst case scenario. The underlying argument is trivial and uses the bivariate margin of a multivariate cdf.

3.5. Estimation of the asymptotic elasticity

Proposition 1(c) brings us to the convenient situation, where the asymptotic elasticity \mathcal{E} can be determined by replacing γ_2 and γ_1 through corresponding estimators. This is accompanied by all advantages of estimating a point, i.e., \mathcal{E} , compared to estimating a curve, i.e., $\mathcal{E}(x_1)$.

Let $\{x_{ij}\}_{i=1}^n$ denote an independently and identically distributed sample of the random variable X_j , $j = 1, 2$. Let the univariate distributions of X_1 and X_2 have tail-monotone densities given by $f_j(x_j; \theta_j)$, $j = 1, 2$. The quasi-log-likelihood function is then determined by $\ell(\theta) = \ell_1(\theta_1) + \ell_2(\theta_2)$, where $\theta = (\theta_1, \theta_2)^\top$ and $\ell_j(\theta_j) = n^{-1} \sum_{i=1}^n \log f_j(x_{ij}; \theta_j)$, $j = 1, 2$. By definition, the quasi Maximum Likelihood (ML) estimator for θ , denoted by $\hat{\theta}$, is the maximizer of the quasi-log-likelihood function $\ell(\theta)$.

The ML-estimator can be shown to be a consistent estimator for the point θ_0 referring to the minimizer of the Kullback-Leibler divergence between the true and assumed density for data generating process, e.g., see [White \(1994, Chapter 3\)](#). The limiting distribution of $\sqrt{n}(\hat{\theta} - \theta_0)$ is also well documented in the literature. For example, [White \(1994, Chapter 6\)](#) demonstrates under suitable regularity conditions that $\sqrt{n}(\hat{\theta} - \theta_0) \rightarrow_{\mathcal{L}} N\{0, \Sigma(\theta_0)\}$, where $\Sigma(\theta_0)$ refers to the robust asymptotic covariance matrix.

Note the important fact that we have not imposed any specific form of dependence between X_1 and X_2 . This is neither required for inference about θ_0 nor for the construction of approximate confidence intervals for the asymptotic elasticity as we will see below. This type of quasi-ML estimation is frequently utilized by practitioners and the first step in the estimation method called inference functions for margins, see, e.g., [Joe and Xu \(1996\)](#); [Joe \(2005\)](#).

Let the tail-exponents γ_1 and γ_2 of the marginal densities be differentiable functions of θ such that $\gamma(\theta) = \{\gamma_1(\theta_1), \gamma_2(\theta_2)\}^\top$. For example, if the parameters $\theta_j = \nu_j$ denote shape parameters of Student's- t_ν distributions, the associated tail-exponents are given by the link function $\gamma_j(\nu_j) = 1 + 1/\nu_j$, $j = 1, 2$. Those link functions are available for several parametric families, e.g., see [Section 3.2](#), and yield a parametric estimator for the asymptotic elasticity

from Proposition 1(c), i.e.,

$$\widehat{\mathcal{E}} = \frac{\gamma_2(\widehat{\theta}_2) - 1}{\gamma_1(\widehat{\theta}_1) - 1}, \quad (6)$$

where $\widehat{\theta} = (\widehat{\theta}_1, \widehat{\theta}_2)^\top$ is the quasi-ML estimator.

The asymptotic elasticity from Proposition 1(c) can also be non-parametrically estimated, for example, using the non-parametric estimator of the tail-exponent from [Holan and McElroy \(2010, Section 3-4\)](#). Yet, the joint asymptotic distribution of non-parametrically estimated $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ cannot be inferred from the results of [Holan and McElroy \(2010\)](#). As a consequence, we are not able to derive approximative confidence intervals for a non-parametrically estimated asymptotic elasticity and focus on the parametric estimator from (6) in the following.

3.6. Approximative confidence intervals

Deriving an approximative distribution of the parametric estimator from (6) is basically an application of the delta-method to $\sqrt{n}(\widehat{\theta} - \theta_0) \rightarrow_{\mathcal{L}} N\{0, \Sigma(\theta_0)\}$ and taking advantage of the fact that the denominator of the estimated asymptotic elasticity is always positive. An approximative distribution is shown in the subsequent result, where $\mathcal{J}\{\gamma(\cdot)\}$ denotes the Jacobian of $\gamma(\cdot)$.

Proposition 2. *Let X_2 and X_1 have tail-monotone densities with tail-exponents $\gamma_2(\theta_{2,0}) \geq 1$ and $\gamma_1(\theta_{1,0}) > 1$ with differentiable $\gamma_j(\cdot)$, $j = 1, 2$. Then, the distribution of $\widehat{\mathcal{E}}$ from (6) can be approximated by*

$$\Phi \left(\frac{\sqrt{n} \mathbf{1}(x)^\top \{\mathbf{1} - \gamma(\theta_0)\}}{[\mathbf{1}(x)^\top \mathcal{J}\{\gamma(\theta_0)\} \Sigma(\theta_0) \mathcal{J}\{\gamma(\theta_0)\}^\top \mathbf{1}(x)]^{1/2}} \right) \quad \text{for large } n,$$

where $\mathbf{1}(x) = (-x, 1)^\top$, $\mathbf{1} = (1, 1)^\top$ and $\Phi(\cdot)$ is the standard Gaussian cdf.

Under the stated assumptions, $\widehat{\mathcal{E}}$ is the ratio of two random variables being jointly Gaussian distributed for large n . Thus, the cdf in Proposition 2 is an approximative distribution. Figure 3.4 shows density functions for the approximative density of $\widehat{\mathcal{E}}$ from (6). Employed parameter combinations of $\gamma_1(\theta_{1,0})$ and $\gamma_2(\theta_{2,0})$ yield the theoretical values $\mathcal{E} = 0.6$ (solid),

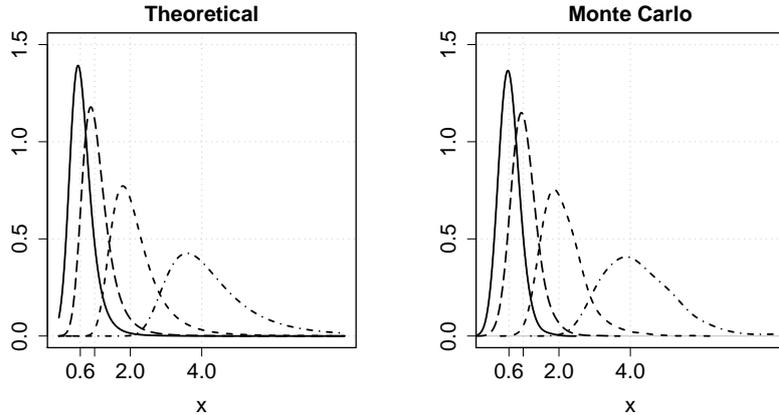


Figure 3.4: The left panel shows the theoretical densities of $\hat{\mathcal{E}}$ from Proposition 2 for $\gamma_2 = 1.15$ (solid), $\gamma_2 = 1.25$ (long-dashed), $\gamma_2 = 1.5$ (dashed) and $\gamma_2 = 2.0$ (dashed-dotted), holding γ_1 fixed at $\gamma_1 = 1.25$. The sample size is $n = 250$ and the covariance matrix coincides with the identity. The right panel illustrates kernel density estimates (bandwidth $h = 0.2$) for estimated asymptotic elasticities following (6). These estimates rely on 1,000 Monte Carlo samples of size $n = 250$ for conditionally tail-dependent X_2 and X_1 . Both risk factors are assumed to follow Student's- t_ν distributions with degrees of freedom $\nu_j = (\gamma_j - 1)^{-1}$, $j = 1, 2$. The tail-exponents γ_2 and γ_1 are chosen as for the left panel and the presented kernel density estimates refer to the corresponding counterparts from the left panel.

$\mathcal{E} = 1$ (long-dashed), $\mathcal{E} = 2$ (dashed) and $\mathcal{E} = 4$ (dashed-dotted). In line with expectations, the densities are rather symmetric around the theoretical value as long as the theoretical value \mathcal{E} is small. Note that the densities become positively skewed for large theoretical values of \mathcal{E} . The exact distribution of the ratio of two jointly Gaussian random variables is given by a Cauchy-type distribution derived in, e.g., Hinkley (1969) and Cedilnik, Kosmelj, and Blejcek (2004). Nonetheless, the proposed cdf suffices to approximate asymptotic confidence intervals for the theoretical values of the asymptotic elasticity \mathcal{E} . This can be inferred from the fact that the kernel density estimates of samples of estimated asymptotic elasticities (right panel of Figure 3.4) are fairly similar to their theoretical counterparts (left panel of Figure 3.4).

4. The subprime mortgage crisis and Lehman Brothers

By interpreting the asymptotic elasticity as a spill-over measure for tail-risk, in the following we analyze the peak period of the subprime mortgage crisis 2007-2008, namely the prominent bankruptcies of Lehman Brothers (LEH) and Bear Stearns (BSC). Hereby, we consider neg-

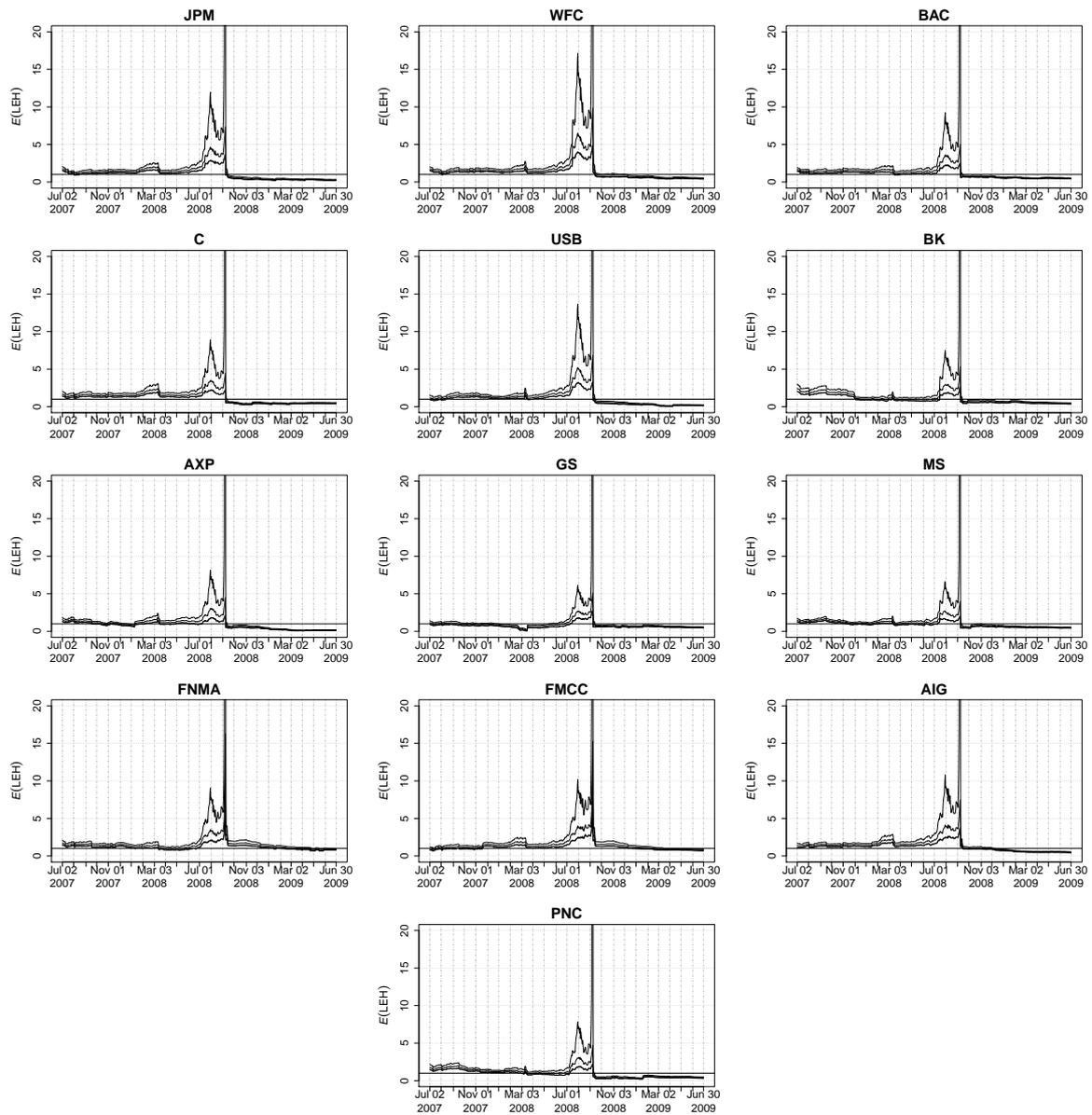


Figure 4.1: Pairwise asymptotic elasticity as measure of spill-over from LEH to JPM, WFC, BAC, C, USB, BK, PNC, AXP, GS, MS, FNMA, FMCC, and AIG for the sample period July 1, 2007 to June 30, 2009. For each panel, the lower solid curve refers to the 5%-quantile and the upper solid curve to the 95%-quantile of the distributional estimate for the asymptotic elasticity.

LEH			BSC		
Date	P_t	$-\log(P_t/P_{t-1})$	Date	P_t	$-\log(P_t/P_{t-1})$
Sep05	16.20	6.57%	Mar10	62.30	11.77%
Sep08	14.15	13.53%	Mar11	62.97	-1.07%
Sep09	7.79	59.69%	Mar12	61.58	2.23%
Sep10	7.25	7.18%	Mar13	57.00	7.73%
Sep11	4.22	54.12%	Mar14	30.00	64.19%
Sep12	3.65	14.51%	Mar17	4.81	183.05%
Sep15	0.21	285.54%	Mar18	5.91	-20.59%

Table 4.1: Price P_t and negative log-return $-\log(P_t/P_{t-1})$ of the stock of LEH and BSC during their most turbulent periods.

ative log-returns of end of the day share prices for 15 major financial institutions in the U.S. In particular, we concentrate on the 13 companies considered in [Diebold and Yilmaz \(2014\)](#), i.e., JPMorgan Chase (JPM), Wells Fargo (WFC), Bank of America (BAC), Citigroup (C), US Bancorp (USB), Bank of New York Mellon (BK), PNC Group (PNC), American Express (AXP), Goldman Sachs (GS), Morgan Stanley (MS), Fannie Mae (FNMA), Freddie Mac (FMCC) and American International Group (AIG). While all of these institutions have survived the turbulent times of the crisis, we enlarge this set of 13 institutions by either LEH or BSC in order to assess the effect from the (near-)bankruptcy of these institutions on the other 13 institutions.

The considered sample period begins in July 1, 2007 and ends in June 30, 2009. As we apply a backward looking rolling window approach with a window length of 250 trading days, our first estimation window begins on August 2, 2006. In other words, data from August 2, 2006 until July 1, 2007 is employed to get an estimate for the considered quantities for July 1, 2007. Then, the window is moved one day forward to estimate the quantities for July 2, 2007, and so on.

Within each window, we filter the univariate time series with a GARCH(1,1) model, see [Engle and Bollerslev \(1986\)](#), and fit Student's- t_ν distributions with $\nu \geq 1$ degrees of freedom to the filtered data. Goodness of fit tests indicate an appropriate description of the data by this approach within each window.² Moreover, this approach yields an estimate $\hat{\nu}$ for each

²Results for fitted GARCH(1,1) models and goodness of fit tests are not reported here, but are available upon request to the authors.

institution at each point in time and permits computing the asymptotic elasticity as well as the considered alternative spill-over measure for tail-risk from the same estimates.

Let k and ℓ represent the institutions, where the tail-risk stems from (ℓ) and is transmitted to (k). We focus particularly on the case when ℓ equals LEH. The asymptotic elasticity can be inferred directly from estimates of the degrees of freedom parameters of the Student's- t_ν distributions, i.e., $\widehat{\mathcal{E}}_{k\ell} = \{\gamma(\hat{\nu}_k) - 1\} / \{\gamma(\hat{\nu}_\ell) - 1\}$ with $\gamma(\nu) = 1 + 1/\nu$.

For comparison, a version of the Δ CoVaR is computed from the same estimates of the rolling window GARCH(1,1) models and fitted Student's- t_ν distributions. [Adrian and Brunnermeier \(2016, Appendix II.A.\)](#) illustrate that Δ CoVaR can be derived from a bivariate Gaussian distribution as

$$\Delta \text{ CoVaR} = \Phi^{-1}(0.95)\sigma_k\rho_{k\ell}, \quad (7)$$

where σ_k denotes the standard deviation of risk factor X_k , $\rho_{k\ell}$ denotes the correlation between the risk factors X_k and X_ℓ and $\Phi^{-1}(0.95)$ is the right 95% quantile of the standard Gaussian distribution. Our robustness checks reveal that replacing the Gaussian quantile in (7) by the estimated Student's- t_ν quantile leads only to insignificant changes of the results. Hereby, the parameter σ_k corresponds to the most recent estimate of the conditional volatility obtained from the GARCH-filter, while also the correlation coefficient $\rho_{k\ell}$, estimated from the filtered data, varies over time due to the rolling window approach.

The pairwise estimates of the asymptotic elasticities for spill-over effects from LEH to the other 13 major financial institutions are presented in [Figure 4.1](#). For each point in time, we report the estimate of the asymptotic elasticity as well as the 5%-quantile (lower solid curve) and the 95%-quantile (upper solid curve) of the distribution for the estimated asymptotic elasticity. Recall that [Proposition 2](#) permits the simple construction of confidence intervals to check whether the true value of the asymptotic elasticity is significantly larger than one. This value serves as a natural reference point because of the interpretation of an elasticity, i.e., LEH can be categorized as risk emitter if the elasticity is larger than one. From a first glance, we observe a very similar pattern for the pairwise asymptotic elasticities for all 13

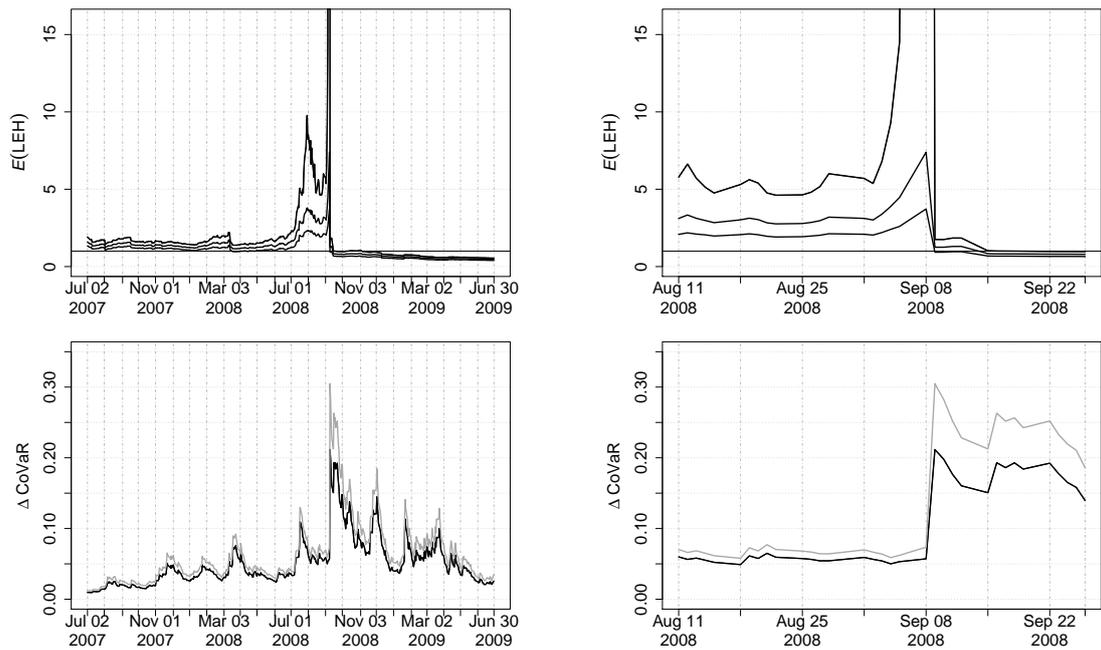


Figure 4.2: Estimated asymptotic elasticities (*upper panels*) and ΔCoVaR (*lower panels*) averaged across all pairwise measures for the 13 financial institutions. The upper left refers to the average asymptotic elasticity for the entire sample period from July 1, 2007 to June 30, 2009, while the upper right panel refers to turbulent period of the LEH crisis from August 11, 2008 to September 26, 2008. The lower solid curves refer to the average 5%-quantile and the upper solid curves to the average 95%-quantile, based on the distributions for the estimated pairwise asymptotic elasticities. Likewise, the lower left and right panel refer to the ΔCoVaR for the entire sample period, and to the turbulent period of the LEH crisis, respectively. The gray lines in the lower panels refer to the ΔCoVaR relying on a Student's- t_ν quantile.

financial institutions. The panels further illustrate the significant increase in the spill-over measure for each institution starting from July 2008 onwards, as well as the spike in the measure in September 2008. Interestingly, there is also a significant drop for the estimated pairwise asymptotic elasticities from mid-September 2008 onwards which we will examine more thoroughly in the following.

As the estimated series of pairwise asymptotic elasticities and Δ CoVaR for all financial institutions show a rather similar pattern, in the following we examine the results for both spill-over measures averaged across the 13 financial institutions.³ Thus, we calculate the average asymptotic elasticity and Δ CoVaR to measure the spill-over effects from LEH to the financial institutions considered in our sample. Results for these average spill-over measures are reported in Figure 4.2. Again, for all panels, the lower solid curve refers to the 5%-quantile and the upper solid curve to the 95%-quantile of the distribution for the estimate of the asymptotic elasticity. Note that also these quantiles have been averaged across the corresponding quantiles for all institutions. Even though this procedure is not entirely correct from a statistical perspective, the results may be interpreted as the spill-over effect from LEH to an average institution.⁴ As the study relies on end-of-the-day prices and returns, both spill-over measures for tail-risk also quantify the transferred risk 'end-of-the-day'.⁵ The gray solid lines in the lower panels of Figure 4.2 refer to the $\widehat{\Delta\text{CoVaR}}$ relying on a Student's- t_ν quantile. For both average asymptotic elasticity and average $\widehat{\Delta\text{CoVaR}}$, the left panels of Figure 4.2 show the estimates of the considered spill-over measures over the entire sample period, while the right panels focus on the most turbulent and volatile weeks of the crisis period from August 11, 2008 to September 26, 2008.

The upper-left panel referring to the asymptotic elasticity reveals one turbulent period before the bankruptcy of LEH. The fact that the lower curve, i.e., the average 5%-quantile, is typically close to one (before July 2008) leads to the conclusion that LEH cannot be categorized

³In favor of a compact presentation of the results, the pairwise estimated Δ CoVaR are not presented but available upon request.

⁴Also recall the very similar behavior of the pairwise asymptotic elasticities and the quantiles of the estimated elasticities reported in Figure 4.1.

⁵Up to the best of our knowledge, methods to backtest estimates of Δ CoVaR are currently under investigation but not available yet.

as a statistically significant risk emitter before July 2008. In other words, before July 2008, under a scenario of extreme financial distress for LEH the spill-over effects from LEH to the other financial institutions in our sample would have been expected to be rather moderate. The status of LEH, however, changes significantly in July 2008, when the estimated average asymptotic elasticity increases substantially and also the average 5%-quantile clearly exceeds the natural reference point of $\mathcal{E} = 1$. Additionally, the pronounced upper curve suggests a potentially large increase in the VaR of the other financial institutions conditional on an extreme shock in the share price of LEH.

Let us now concentrate on the trading week September 8-12, 2008. As illustrated in the upper-right panel of Figure 4.2, the asymptotic elasticity has a dramatic drop on September 9 and remains on that level for the remainder of the week. We interpret and explain this fall below, after presenting some background information. As documented in Table 4.1, the negative log-return of LEH was about 60% on September 9, due to a drop in the stock price from \$14.15 to \$7.79. On September 10, LEH publicly announced that a massive loss of \$3.9 billion had to be expected for the third quarter of 2008. Hence, September 10 can be considered as the last point in time for market participants to realize that LEH is in financial distress. This might also lead to the following conclusion: If all market participants know that LEH is in distress, there is only little uncertainty left which can be transferred from LEH to other market participants. The calculated average asymptotic elasticity clearly portrays this new market situation and exhibits a significant drop after September 9, 2008. Finally, on September 15, LEH officially filed for bankruptcy protection.

As illustrated in the lower-left panel of Figure 4.2, the Δ CoVaR measure does not identify the turbulent period related to the LEH crisis as precisely as the asymptotic elasticity. Furthermore, the lack of a reference value and confidence intervals for Δ CoVaR make it more difficult to specify an appropriate point in time, when LEH started to become systemically relevant. Choosing a different quantile than 95% for Δ CoVaR, or even a different quantile function than $\Phi^{-1}(\cdot)$, is not a remedy, as the categorization of LEH as a risk emitter or receiver is unaffected by the position of the curve. A different quantile-level merely shifts the curve up or down. Similar to our analysis for the average asymptotic elasticity, the lower-right

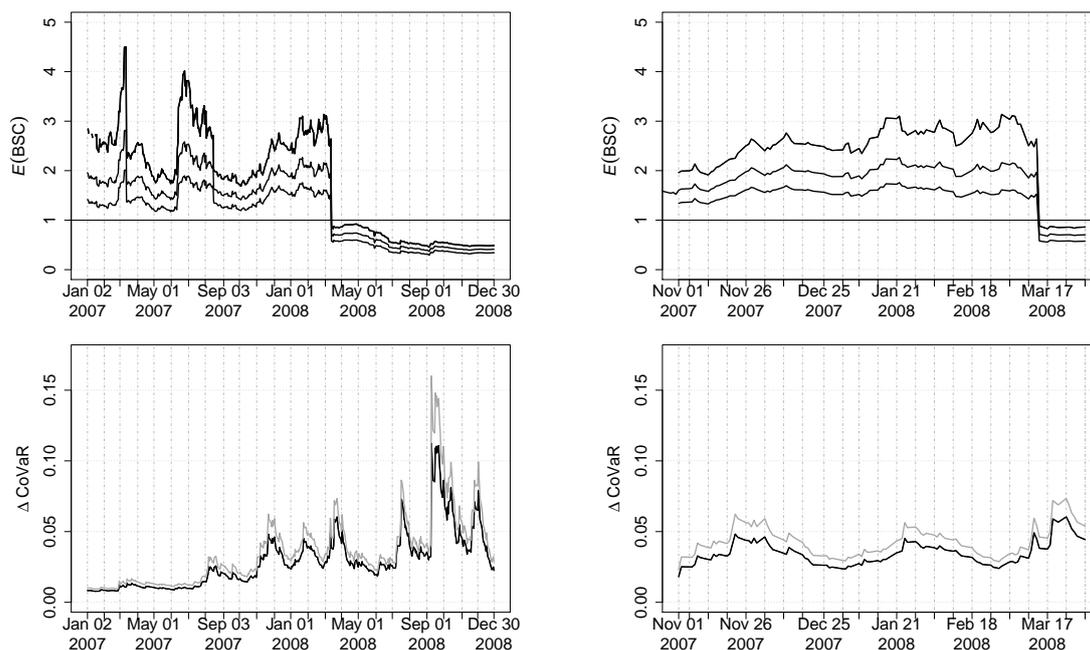


Figure 4.3: Estimated asymptotic elasticities (*upper panels*) and Δ CoVaR (*lower panels*) averaged across all pairwise measures for the 13 financial institutions. The upper left refers to the average asymptotic elasticity for the sample period from January 2, 2007 to December 30, 2008, while the upper right panel refers to turbulent period prior and post the BSC bankruptcy from November 1, 2007 to March 31, 2008. The lower solid curve refers to the average 5%-quantile and the upper solid curve to the average 95%-quantile, based on the distributions for the estimated pairwise asymptotic elasticities. Likewise, the lower left and right panel refer to the Δ CoVaR for the entire sample period, and to the turbulent period around the near-collapse of BSC, respectively. The gray lines in the lower panels refer to the Δ CoVaR relying on a Student's- t_ν quantile.

panel of Figure 4.2 zooms into the turbulent period prior and after the bankruptcy of LEH. In contrast to our spill-over measure for tail-risk, the $\widehat{\Delta\text{CoVaR}}$ peaks on September 9 and remains on about that level until the end of September. This finding is not easy to interpret from an economic point of view, since the share price of LEH is at a penny-stock level from September 15 onwards. Therefore, a further transfer of risk from LEH to the other financial institutions through a further drop in the share price of LEH is rather unlikely.

We also conduct a similar analysis for the period prior and after the near-bankruptcy and takeover of Bear Stearns (BSC). Similar to the analysis for LEH, we are interested in measuring spill-over of tail-risk from BSC to the other 13 financial institutions. Figure 4.3 provides a plot of the estimated average asymptotic elasticity as well as the average Δ CoVaR for the

entire sample period and for the period prior to the near-collapse of BSC and take-over by JPMorgan Chase. Likewise, Table 4.1 shows the course of the stock price of BSC for a number of days in March 2008, when the bank suffered from illiquidity and was under serious financial distress. Even though the economic reasons causing the near-collapse of BSC differ from those causing the default of LEH, the presented line of statistical arguments for the analysis of the LEH-crisis as well as the statistical interpretation can be transferred to the BSC-crisis. It should be noticed though that the spill-over effect from BSC is - on average - not as strong as in the case of LEH.

Remark 4. *The drop in the asymptotic elasticity on September 9, 2008, does not coincide with the end of the crisis caused by LEH. For instance, the strong shock on September 11, see Table 4.1, has just less impact on the tail-risk of the other institutions than the shock on September 9. This is due to a higher probability mass in the estimated right tail of the distribution of LEH, when the estimation window is moved two days ahead and includes the extreme returns for LEH on September 8 and 9. Furthermore, note that the asymptotic elasticity does not provide any information about the propagation of the shock from September 9. The propagation of shocks has to be analyzed by different methods, e.g., White, Kim, and Manganelli (2015) propose vectorautoregressions for quantiles in order to study dynamics of tail-events over time.*

Remark 5. *Recall the importance of the relation of marginal distributions to each other, i.e., the asymptotic elasticity is the ratio of the tail-exponents (minus one) having LEH in the denominator and the respective other institution in the numerator. While the tail-exponent of LEH was frequently the lowest among all financial institutions before the critical September 9, 2008, the tail-exponents of other institutions have increased enormously before September 9. This is true, for example, for FNMA, FMCC and AIG. These simple statistical facts tell us that the collapse of LEH was like removing a stable segment from a network that was already under stress. This situation is also graphically illustrated in the left panel of Figure 4.4 showing the estimated tail-exponent for LEH and the average over the estimated tail-exponents for all remaining institutions. The black line referring to the tail-exponent of LEH goes down during spring 2008 while the gray line stays almost on the same level. Similar conclusions*

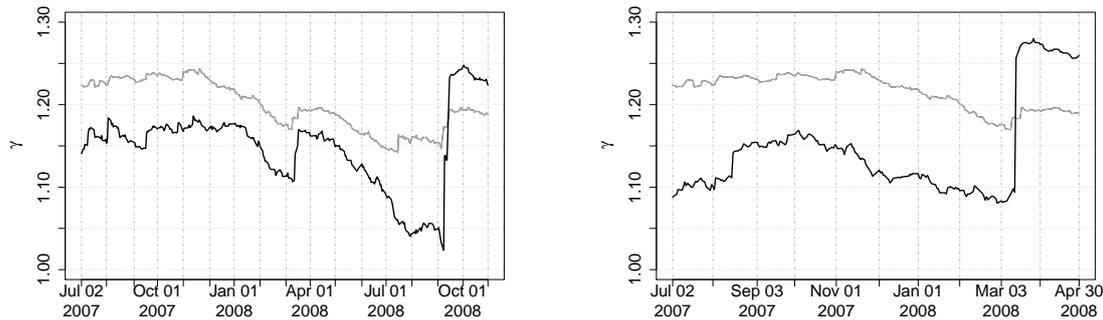


Figure 4.4: Estimated tail-exponents: the gray lines in both panels refer to the average estimated tail-exponents across the 13 financial institutions for the period July 2, 2007 to October 31, 2008 (*left panel*) and the period July 2, 2007 to April 30, 2008 (*right panel*). The estimated tail-exponent for LEH (*left panel*) and BSC (*right panel*) are depicted as black lines in the left and right panel respectively.

can also be drawn for the near-collapse of BSC in March 2008, where estimated tail exponents for the 13 financial institutions and BSC are displayed in the right panel of Figure 4.4.

5. The European sovereign debt crisis

This case study relies on five-year maturity credit default swap (CDS) contracts traded on their reference bonds for ten countries in the Euro-zone, namely Austria (AUT), Belgium (BEL), Germany (GER), Spain (ESP), France (FRA), Greece (GRC), Ireland (IRL), Italy (ITA), Netherlands (NLD) and Portugal (PRT). The corresponding CDS spreads are available for these countries at a daily frequency for the sampling period February 2009 until December 2012 and provided by Bloomberg.

Lucas *et al.* (2014) use a flexible econometric model to assess the probability of a conditional sovereign default from observed CDS prices. The approach relies on a dynamic skew-Student's- t distribution which reasonably describes stylized facts of changes in CDS prices, e.g., skewness, long-tails as well as dynamic volatilities and correlations. The model is applied to price changes in spreads of CDS contracts ensuring the holder against the default event of countries from the Euro-area during the period of the Euro-area sovereign debt crisis. Among others, the authors investigate implications from a credit default event of GRC.

Our approach is complementary to the analysis of Lucas *et al.* (2014, Section 3.5) in the sense

that the asymptotic elasticity quantifies effects from a possible default event on a quantile-level. In contrast, the approach of [Lucas *et al.* \(2014\)](#) quantifies the default probability given a credit default of another country. As in [Section 4](#), we use a rolling window of length 250 trading days to model each univariate time series of changes in the prices of the CDS spreads. Within each window, Student's- t_ν distributions are fit to filtered data obtained from GARCH(1,1) models. Employing Student's- t_ν distributions for the margins is close to the specification of [Lucas *et al.* \(2014\)](#), because the marginal distributions of the underlying multivariate skew-Student's- t distribution have tail-exponents that are identical to those of an ordinary Student's- t_ν distribution. Recall, however, that our approach does not require a specification of the dependence between the univariate time series.

[Figure 5.1](#) illustrates the asymptotic elasticities for the considered Euro-countries given a hypothetical credit event of GRC, i.e., the CDS price for Greek sovereigns takes an extremely large value. Overall, the asymptotic elasticities are below one which indicates that a credit event in GRC would not have led to a striking increase in the tail-risk of the other differenced CDS price series. In other words, the tails of the distributions of changes in CDS prices for the considered nine Eurozone countries are fairly robust with respect to a default in GRC. Furthermore, most asymptotic elasticities decrease since January 2011 which is caused by an increase in the price of the CDS spread for GRC by a factor of ten over 2011. This reduction in fallout for the other Euro-countries is also observed in [Lucas *et al.* \(2014\)](#) on the basis of conditional probabilities. They, additionally, point out the consistency of such findings with the behavior of market participants who prepare for the possibility of a credit default as it becomes more likely to occur. The less pronounced decline in the asymptotic elasticities for IRL and PRT is caused by the fact that the tail-exponents of IRL and PRT are more similar to the tail-exponent of GRC in comparison to the tail-exponents of the other countries. Note that IRL and PRT were also supported through an EU/IMF program during 2011.

6. Conclusion

In this work we have proposed a novel approach to quantify effects of an extreme outcome in

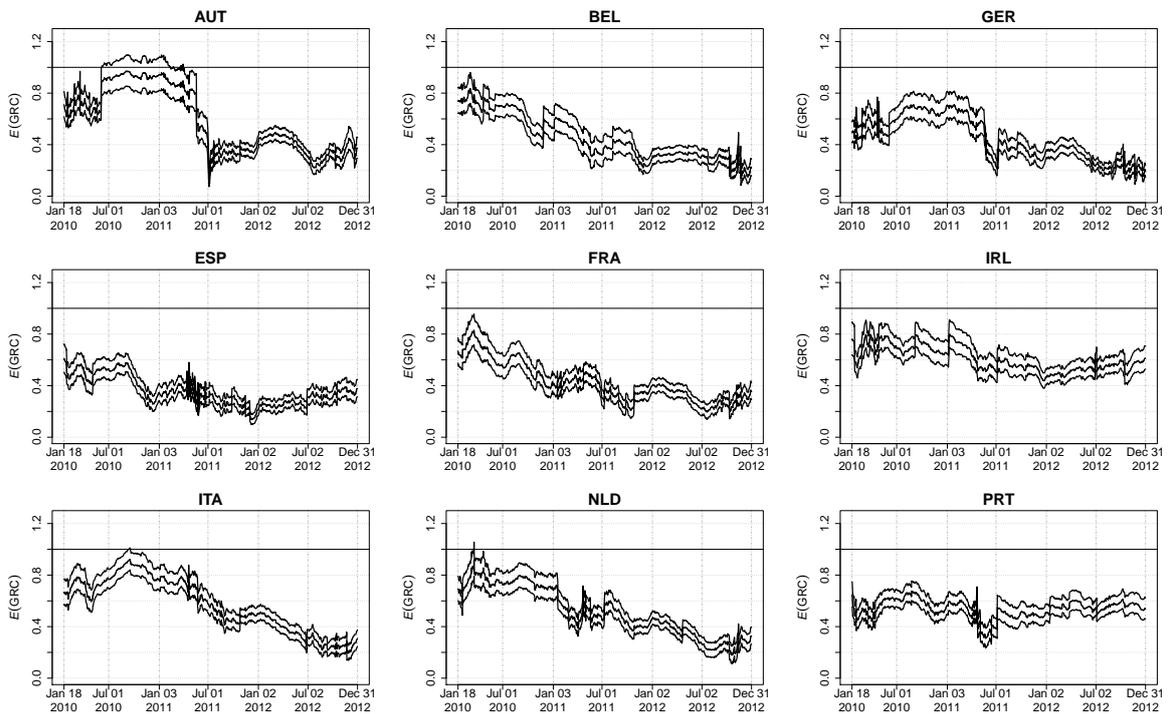


Figure 5.1: Estimated pairwise asymptotic elasticity as measure of spill-over from GRC to AUT, BEL, GER, ESP, FRA, IRL, ITA, NLD, and PRT for the sample period January 18, 2010 to December 31, 2012. For each panel, the lower solid curve refers to the 5%-quantile and the upper solid curve to the 95%-quantile of the distribution of the estimated asymptotic elasticity.

an explanatory variable on a dependent variable via the asymptotic elasticity of a conditional quantile, linking the dependent and explanatory variable. The asymptotic elasticity is an ordinary elasticity but its argument takes the value of an extreme event. Hence, the intuitive interpretation of the ordinary elasticity can be transferred to the asymptotic elasticity. A closed form expression for the asymptotic elasticity is presented which is independent of the exact relation between explanatory and dependent variable. A simple parametric estimation and inference approach is presented.

The benefit of the proposed framework is demonstrated in two empirical examples related to the literature on financial economics, in particular the measurement of systemic risk. By interpreting the asymptotic elasticity as a spill-over measure for tail-risk, the estimation results reveal statistically significant spill-over effects for tail-risk from Lehman Brothers as well as Bear Stearns to other financial institutions during the subprime mortgage crisis. We also find that the calculated spill-over measures show a significant increase before the default of these institutions, while the measures drop substantially after the news of financial distress of these institutions became public, making their share price drop enormously. We interpret these outcomes in the following way: If all market participants are already aware that a financial institution is in distress, there is only little uncertainty left which can be transferred from this institution to other market participants. Moreover, the proposed methodology has confirmed earlier empirical findings related to the European sovereign debt crisis. In particular, our results suggest that based on the dynamic behavior of sovereign CDS spreads, a default of Greece would not have had significant impacts on the solvency of other countries in the Euro-area.

A. Technical appendix

This appendix is chronologically structured as Section 3: First of all, tail-monotone density functions, an analytical tractable form for the elasticity and conditional tail-(in)dependence are discussed. Secondly, Proposition 1 is proven. Thirdly, Figures 3.2 and 3.3 are revisited in the sense that we partly verify underlying regularity assumptions for these examples. Fourthly, Proposition 2 is briefly proven. Last but not least, the discussion on more than one explanatory variable is formally inspected.

Tail-monotone densities

Let $Q(u)$ be the quantile function of a probability law with quantile density $q(u) = Q'(u)$, cdf $F(x)$ and density $f(x) = F'(x)$. Equivalent to Definition 1, the tail-exponent $\gamma > 0$ is defined as $\gamma = \lim_{u \rightarrow 1} \log f\{Q(u)\} / \log(1 - u)$, where (i) $\gamma < 1$, (ii) $\gamma = 1$ and (iii) $\gamma > 1$ characterizes (i) short-, (ii) exponential- and (iii) long tails. Holan and McElroy (2010, Section 4) argue that the tail-control requirement of Definition 1, i.e., $\sup_{x \in (a,b)} F(x)\{1 - F(x)\}|f'(x)|/f(x)^2 \leq \gamma$, is in this definition of the tail-exponent automatically fulfilled. Thus, the assumption of tail-monotonicity does not seem to be restrictive for our modeling purpose.

Parzen (1979, Section 9) shows for the class of tail-monotone densities that $f\{Q(u)\} \sim (1-u)^\gamma$, $u \rightarrow 1$, by relying on a general approximation of the tail-area of a distribution, see Andrews (1973). Since $f\{Q(u)\} = 1/q(u)$, the result of Parzen (1979) translates immediately into $q(u) \sim (1-u)^{-\gamma}$, $u \rightarrow 1$.

Alternative representation for \mathcal{E}

Following Sklar (1959), $F(x_1, x_2)$ can be decomposed into the marginal cdfs and a copula function $C(\cdot)$ describing the dependence between random variables X_1 and X_2 such that $F(x_1, x_2) = C\{F_1(x_1), F_2(x_2)\}$. Overviews of copulae are given in Joe (1997) and Nelsen (2006), while recent developments for mathematical and quantitative finance are presented in Jaworski, Durante, and Härdle (2013). Based on $U_j = F_j(X_j)$ and $u_j = F_j(x_j)$ with $U_j \sim U(0, 1)$ and $u_j \in (0, 1)$, $j = 1, 2$, an alternative representation of the conditional cdf (1)

is given by

$$C_{U_2|U_1=u_1}(u_2) = P(U_2 \leq u_2 | U_1 = u_1). \quad (8)$$

The latter can be derived from the copula function $C(\cdot)$ but is not a copula for itself. The inverse of $C_{\cdot|\cdot}(u_2)$, denoted by $C_{\cdot|\cdot}^{-1}(v)$, $v \in (0, 1)$, is called C-quantile and introduced in [Bouyé and Salmon \(2009\)](#). Based on the C-quantile and $Q_j(u_j)$, $u_j \in (0, 1)$, $j = 1, 2$, the conditional quantile from (2) can be rewritten as

$$\begin{aligned} Q_{X_2|X_1=x_1}(\alpha) &= Q_2\{C_{F_2(X_2)|F_1(X_1)=F_1(x_1)}^{-1}(\alpha)\} \\ &= Q_2\{C_{U_2|U_1=u_1}^{-1}(\alpha)\}. \end{aligned} \quad (9)$$

The C-quantile representation of $\mathcal{E}(x_1)$ relies on the derivative of $Q_{X_2|X_1=x_1}(\alpha)$ wrt x_1 . To analyze this derivative analytically, let $q_j(u_j) = Q'_j(u_j)$, $u_j \in (0, 1)$, be the unconditional quantile density, $j = 1, 2$. Based on (9), the derivative of $Q_{X_2|X_1=x_1}(\alpha)$ wrt x_1 is

$$\frac{\partial}{\partial x_1} Q_{X_2|X_1=x_1}(\alpha) = \frac{q_2\{C_{U_2|U_1=u_1}^{-1}(\alpha)\}}{q_1(u_1)} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha). \quad (10)$$

Equation 10 demonstrates how the partial derivative of the conditional quantile wrt to x_1 is disentangled into components of the distribution of the dependent variable (through $q_2(\cdot)$), the explanatory variable (through $q_1(\cdot)$) and their dependence (through $C_{U_2|U_1=u_1}^{-1}(\alpha)$ and $\frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha)$). Using (10), we obtain an alternative expression for the asymptotic elasticity

$$\mathcal{E} = \lim_{u_1 \rightarrow 1} \frac{Q_1(u_1)q_2\{C_{U_2|U_1=u_1}^{-1}(\alpha)\}}{q_1(u_1)Q_2\{C_{U_2|U_1=u_1}^{-1}(\alpha)\}} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha), \quad (11)$$

which is analytically more tractable than the limit of (3). Moreover, (11) immediately reveals that any scaling (or weighting) of the risk factors X_2 and X_1 vanishes. The C-quantile is not affected by scaling either, since copulas describe dependence on a quantile-level and are invariant under strictly increasing transformations of the risk factors X_1 and X_2 , see [Nelsen](#)

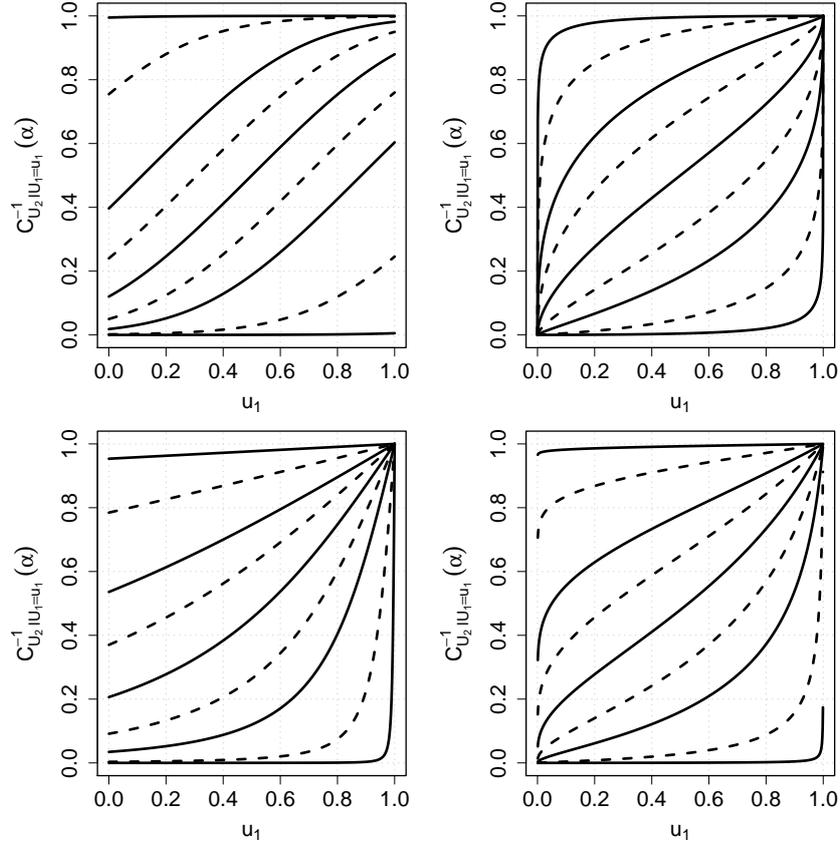


Figure A.1: C-quantile functions $C_{U_2|U_1=u_1}^{-1}(\alpha)$ for the bivariate Frank (upper-left), Gaussian (upper-right), survival Clayton (lower-left) and Gumbel copula (lower-right) for a parameter referring to Kendall's $\tau = 1/2$. The alternating lines (solid and dashed) refer to $\alpha \in \{0.0001, 0.01, 0.1, 0.25, 0.5, 0.75, 0.9, 0.99, 0.9999\}$ - bottom-up ordered.

(2006, Theorem 2.4.3).

Conditional tail-(in)dependence

According to Definition 2, X_2 and X_1 are conditionally tail-independent, if $Q_{X_2|X_1=x_1}(\alpha) \sim g(\alpha)$, $x_1 \rightarrow \infty$. This definition is equivalent to $C_{U_2|U_1=u_1}^{-1}(\alpha) \sim F_2\{g(\alpha)\}$, $u_1 \rightarrow 1$. It should be noticed that this notion of conditional tail-independence is not equivalent to the generic definition of tail-independence based on the tail-dependence coefficient, see Bernard and Czado (2015, Section 4). Figure A.1 presents four sets of C-quantile curves derived from parametric copulas. For these copulas, there exists a mapping between the parameter of the underlying copula and Kendall's τ measuring the strength of dependence in a non-parametric

way. The copula parameters employed below are chosen to refer to $\tau = 1/2$.

The formula for the C-quantile curves of the Frank copula presented in the upper-left panel of Figure A.1 is given by

$$C_{U_2|U_1=u_1}^{-1}(\alpha; \delta) = -\frac{1}{\delta} \log \left[1 - \frac{\alpha \{1 - \exp(-\delta)\}}{\exp(-\delta u_1) + \alpha \{1 - \exp(-\delta u_1)\}} \right] \quad \text{with } \delta \in \mathbb{R} \setminus \{0\}, \quad (12)$$

where Figure A.1 uses $\delta \approx 5.74$. As the quantile curves do not converge towards one as $u_1 \rightarrow 1$, the C-quantile implied by the Frank copula exhibits conditional tail-independence, see Bernard and Czado (2015, Proposition 4.3). Moreover, the derivative of (12) wrt u_1 can be straightforwardly shown to be zero as $u_1 \rightarrow 1$. In other words, the value of the implied conditional quantile $Q_{X_2|X_1=x_1}(\alpha)$ does not change if x_1 is increased irrespective of the employed marginal distributions for X_1 and X_2 . The formula for the C-quantile curves of the Gaussian copula presented in the upper-right panel of Figure A.1 is given by

$$C_{U_2|U_1=u_1}^{-1}(\alpha; \rho) = \Phi \left\{ \Phi^{-1}(\alpha) \sqrt{1 - \rho^2} + \rho \Phi^{-1}(u_1) \right\} \quad \text{with } \rho \in (-1, 1). \quad (13)$$

The C-quantile in Figure A.1 relies on $\rho \approx 0.71$ and approaches one as $u_1 \rightarrow 1$. Thus, the C-quantile of the Gaussian copula does not exhibit conditional tail-independence.

Our notion of conditional tail-dependence in the right tail relies on asymptotic properties of the conditional quantile, i.e., $Q_{X_2|X_1=x_1}(\alpha) \rightarrow \infty$ as $x_1 \rightarrow \infty$ such that $\frac{\partial}{\partial x_1} Q_{X_2|X_1=x_1}(\alpha)$ is positive and bounded on an interval (x_0, ∞) for some x_0 . These requirements rule out conditional tail-independence as $x_1 \rightarrow \infty$ and ensure that the conditional quantile does not converge too fast to infinity. The equivalent properties in terms of the C-quantile are given by $C_{U_2|U_1=u_1}^{-1}(\alpha) \rightarrow 1$ as $u_1 \rightarrow 1$ such that $\frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha)$ is positive and bounded on an interval $(u_0, 1)$ for $u_0 = F_1(x_0)$. The latter equivalence, however, is only true if the densities $f_1(x_1)$ and $f_2(x_2)$ are strictly positive on the considered support. To see this, we refer to (10) where $q_j(\cdot)$ must be replaced by $1/f_j\{Q_j(\cdot)\}$, $j = 1, 2$.

Denoting the density function of the standard Gaussian distribution by $\phi(\cdot)$ permits writing

the derivative of the C-quantile of the Gaussian copula wrt u_1 as

$$\frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha; \rho) = \rho \frac{\phi \left\{ \Phi^{-1}(\alpha) \sqrt{1 - \rho^2} + \rho \Phi^{-1}(u_1) \right\}}{\phi \left\{ \Phi^{-1}(u_1) \right\}}, \quad (14)$$

which can be shown to converge to infinity as $u_1 \rightarrow 1$. Therefore, the C-quantile of the Gaussian copula fails to describe conditional tail-dependence. Figure A.1 also reveals that the derivative of the C-quantile of the Gaussian copula is unbounded as $u_1 \rightarrow 1$.

Any C-quantile of the survival Clayton copula supports conditional tail-dependence in the right tail, since the C-quantile converges to one and the derivative is positive as well as bounded. To verify that the derivative of the C-quantile derived from the rotated Clayton copula is bounded, note that it admits the representation

$$C_{U_2|U_1=u_1}^{-1}(\alpha; \delta) = 1 - \left[\left\{ (1 - \alpha)^{-\frac{\delta}{1+\delta}} - 1 \right\} (1 - u_1)^{-\delta} + 1 \right]^{-\frac{1}{\delta}} \quad \text{with } \delta \in \mathbb{R}_+ \setminus \{0\}. \quad (15)$$

The derivative of (15) wrt u_1 can be calculated as

$$\begin{aligned} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha; \delta) &= (1 - u_1)^{-(1+\delta)} \left\{ (1 - \alpha)^{-\frac{\delta}{1+\delta}} - 1 \right\} \\ &\quad \cdot \left[1 + (1 - u_1)^{-\delta} \left\{ (1 - \alpha)^{-\frac{\delta}{1+\delta}} - 1 \right\} \right]^{-\frac{1+\delta}{\delta}}, \end{aligned} \quad (16)$$

with corresponding limit

$$\lim_{u_1 \rightarrow 1} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha; \delta) = \left\{ (1 - \alpha)^{-\frac{\delta}{1+\delta}} - 1 \right\}^{-\frac{1}{\delta}}, \quad (17)$$

which is obviously bounded for $\alpha \in (0, 1)$. This is graphically demonstrated in the lower-left panel of Figure A.1 which is based on $\delta = 2$.

The implied C-quantile of the Gumbel copula exhibits conditional tail-dependence only for some values of α which is shown in the lower-right panel of Figure A.1. An analytical formula

for this C-quantile is given by

$$C_{U_2|U_1=u_1}^{-1}(\alpha; \delta) = \exp \left[- \left\{ - (-\log u_1)^\delta + \left((\delta - 1) W_0 \left[\frac{\{-\alpha u_1 (-\log u_1)^{-\delta} \log u_1\}^{-\frac{1}{\delta-1}}}{\delta - 1} \right] \right)^\delta \right\}^{\frac{1}{\delta}} \right] \quad \text{with } \delta > 1.$$

where $W_0(x)$ denotes the upper branch of the Lambert-W function. The derivative of $C_{U_2|U_1=u_1}^{-1}(\alpha; \delta)$ wrt u_1 has an untractable form and is, thus, not presented here. Furthermore, this derivative cannot be shown to be bounded for all $\alpha \in (0, 1)$. As illustrated in the lower-right panel of Figure A.1 which uses $\delta = 2$, the derivatives are not bounded in the right tail for too small values of $\alpha \in (0, 1)$ as $u_1 \rightarrow 1$. Nonetheless, numerical experiments can be used to show that $\frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha; \delta)$ is positive and bounded, as $u_1 \rightarrow 1$, for $\delta \gg 1$ and $\alpha \gg 1/2$.

Proof of Proposition 1

Let $Q(u)$, $u \in (0, 1)$, be the quantile function for random variable $X \in \mathbb{R}$ having a tail-monotone density function. If $Q(u) \rightarrow \infty$ as $u \rightarrow 1$, there is a function $a(u)$ satisfying $Q(u) \sim a(u)$, $u \rightarrow 1$. This follows from the fact $q(u) = Q'(u) \sim (1 - u)^{-\gamma}$, $u \rightarrow 1$. Without loss of generality, we can work with the centered random variable $\bar{X} = X - E(X)$ such that $E(\bar{X}) = 0$. As a consequence, the constant of integration K of the indefinite integral $\int q(u) du = Q(u) + K$ can be assumed to be zero, i.e., $K = 0$. Therefore, $a(u)$ can be defined by $a(u) = (\gamma - 1)^{-1}(1 - u)^{1-\gamma}$ in case $\gamma > 1$ and by $a(u) = -\log(1 - u)$ in case $\gamma = 1$. Note that $Q(u) = -\log(1 - u)$ is the quantile function of the standard Exponential distribution.

To prove part (c) for $\gamma_1, \gamma_2 > 1$, recall that the asymptotic elasticity \mathcal{E} can be expressed in terms of $u_1 = F_1(x_1) \in (0, 1)$ according to (11), so that

$$\begin{aligned} \mathcal{E}(u_1) &= \frac{\gamma_2 - 1}{\gamma_1 - 1} \frac{(1 - u_1)^{1-\gamma_1} \{1 - C_{U_2|U_1=u_1}^{-1}(\alpha)\}^{-\gamma_2}}{(1 - u_1)^{-\gamma_1} \{1 - C_{U_2|U_1=u_1}^{-1}(\alpha)\}^{1-\gamma_2}} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha) \\ &= \frac{\gamma_2 - 1}{\gamma_1 - 1} \frac{1 - u_1}{1 - C_{U_2|U_1=u_1}^{-1}(\alpha)} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha) \quad \text{as } u_1 \rightarrow 1. \end{aligned} \quad (18)$$

Since $\frac{\partial}{\partial x_1} Q_{X_2|X_1=x_1}(\alpha)$ is positive and bounded on (x_0, ∞) for some x_0 , $\frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha)$ is positive and bounded on $(u_0, 1)$, $u_0 = F_1(x_0)$, as well. As the limit (18) is not well defined, we apply l'Hôpital's rule to the middle part to obtain

$$\mathcal{E}(u_1) = \frac{\gamma_2 - 1}{\gamma_1 - 1} \frac{1}{\frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha)} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha) = \frac{\gamma_2 - 1}{\gamma_1 - 1} \quad \text{as } u_1 \rightarrow 1.$$

Now, let $\gamma_1 > 1$ and $\gamma_2 = 1$ to obtain

$$\begin{aligned} \mathcal{E}(u_1) &= \frac{1}{1 - \gamma_1} \frac{(1 - u_1)^{1-\gamma_1} \{1 - C_{U_2|U_1=u_1}^{-1}(\alpha)\}^{-1}}{(1 - u_1)^{-\gamma_1} \log\{1 - C_{U_2|U_1=u_1}^{-1}(\alpha)\}} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha) \\ &= \frac{1}{1 - \gamma_1} \frac{1 - u_1}{\{1 - C_{U_2|U_1=u_1}^{-1}(\alpha)\} \log\{1 - C_{U_2|U_1=u_1}^{-1}(\alpha)\}} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha) \quad \text{as } u_1 \rightarrow 1. \end{aligned} \quad (19)$$

L'Hôpital's rule yields

$$\mathcal{E}(u_1) = \frac{(1 - \gamma_1)^{-1}}{[1 + \log\{1 - C_{U_2|U_1=u_1}^{-1}(\alpha)\}] \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha)} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha) \quad \text{as } u_1 \rightarrow 1.$$

As $C_{U_2|U_1=u_1}^{-1}(\alpha) \rightarrow 1$ as $u_1 \rightarrow 1$ under conditional tail-dependence, $\mathcal{E}(u_1) = 0$ as $u_1 \rightarrow 1$.

To prove part (b), instead of (19), we obtain

$$\mathcal{E}(u_1) = (1 - \gamma_2) \frac{(1 - u_1) \log(1 - u_1)}{1 - C_{U_2|U_1=u_1}^{-1}(\alpha)} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha) \quad \text{as } u_1 \rightarrow 1. \quad (20)$$

Since the limit of (20) is not well defined, we apply l'Hôpital's rule and get

$$\mathcal{E}(u_1) = (1 - \gamma_2) \frac{1 + \log(1 - u_1)}{\frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha)} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha) = \infty \quad \text{as } u_1 \rightarrow 1.$$

To prove part (a) for $\gamma_2, \gamma_1 > 1$, note that $C_{U_2|U_1=u_1}^{-1}(\alpha) \sim F_2\{g(\alpha)\}$, $u_1 \rightarrow 1$, by the definition of conditional tail-independence, so that (18) can be rewritten

$$\mathcal{E}(u_1) = \frac{\gamma_2 - 1}{\gamma_1 - 1} \frac{1 - u_1}{1 - F_2\{g(\alpha)\}} \frac{\partial}{\partial u_1} F_2\{g(\alpha)\} = 0 \quad \text{as } u_1 \rightarrow 1.$$

The other three cases when (i) $\gamma_2 = \gamma_1 = 1$, (ii) $\gamma_2 > 1$, $\gamma_1 = 1$ and (iii) $\gamma_2 = 1$, $\gamma_1 > 1$ follow

along similar lines. □

Figures 3.2 and 3.3 revisited

In the examples of Figures 3.2 and 3.3, the conditional quantiles are built from the C-quantile representation given in (9). The marginal distributions include the cdfs of the standard Gaussian, Cauchy, Pareto and Student's- t laws. Among these, only the quantile density and tail-exponent of the Student's- t distribution are not documented in the literature yet. The others are reported in Parzen (1979). In the subsequent paragraphs, we verify that the tail-exponent of the Student's- t_ν density equals $\gamma = 1 + 1/\nu$ and show that the C-quantile $C_{U_2|U_1=u_1}^{-1}(\alpha)$ implied by the Cauchy-type copula has non-zero and bounded derivatives as $u_1 \rightarrow 1$. Furthermore, we present the derivative for the C-quantile implied by the Student's- t copula. The requirements that the derivative of the corresponding C-quantile is bounded might depend on the parameters and value of $\alpha \in (0, 1)$ though. Properties of the C-quantiles (and their derivatives) related to the Frank- and survival Clayton copula have been discussed above. The corresponding graphics in Figure 3.2 and 3.3 use parameters $\delta = 18.19$ and $\delta = 1$, which refer to values of Kendall's τ of $\tau = 4/5$ and $\tau = 1/3$ respectively.

Denote by $I_z(a, b)$ the regularized incomplete beta function. Solving $s = I_z(x, y)$ wrt z for $s \in (0, 1)$, gives the inverse of the regularized incomplete beta function denoted by $I_s^{-1}(a, b)$. Elementary calculations determine the quantile density of a Student's- t_ν distribution by

$$q(u; \nu) = \sqrt{\nu} B\left(\frac{\nu}{2}, \frac{1}{2}\right) I_{v(u)}^{-1}\left(\frac{\nu}{2}, \frac{1}{2}\right)^{-\frac{1+\nu}{2}} \quad \text{for} \quad v(u) = \begin{cases} 2(1-u) & \text{if } u \in (1/2, 1) \\ 2u & \text{if } u \in (0, 1/2) \\ 1 & \text{if } u = 1/2 \end{cases}, \quad (21)$$

with beta function $B(\cdot, \cdot)$. From the definition of the tail-exponent we get $\gamma = 1 + 1/\nu$, i.e.,

$$\gamma = \lim_{u \rightarrow 1} \frac{\log f\{Q(u; \nu); \nu\}}{\log(1-u)} = \lim_{x \rightarrow \infty} \frac{\log f(x; \nu)}{\log\{1 - F(x; \nu)\}} = 1 + \frac{1}{\nu}.$$

For Cauchy-distributed X_2 and X_1 , i.e., $X_j \sim \text{Cauchy}(\mu_j, \sigma_j)$, $j = 1, 2$, Bernard and Czado (2015, Section 3.3) derive a conditional cdf under the assumption that the quantile of X_2

conditional on X_1 is linear and $\sigma_2 > |b|\sigma_1$ for dependence parameter $b \in (-1, 1)$. The implied dependence is called ‘‘Cauchy copula’’ whose C-quantile is determined by

$$C_{U_2|U_1=u_1}^{-1}(\alpha; \sigma_1, \sigma_2, b) = \frac{1}{2} + \frac{1}{\pi} \arctan \left[\frac{\cot(\alpha\pi)(\sigma_1|b| - \sigma_2) + \sigma_1 b \tan \left\{ \pi \left(u_1 - \frac{1}{2} \right) \right\}}{\sigma_2} \right].$$

The derivative of $C_{U_2|U_1=u_1}^{-1}(\alpha; \sigma_1, \sigma_2, b)$ wrt u_1 can be explicitly computed as

$$\frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha; \sigma_1, \sigma_2, b) = \frac{\sigma_1 b \sec \left\{ \pi \left(u_1 - \frac{1}{2} \right) \right\}^2}{\sigma_2 + \sigma_2^{-1} \left[(\sigma_1|b| - \sigma_2) \cot(\alpha\pi) + \sigma_1 b \tan \left\{ \pi \left(u_1 - \frac{1}{2} \right) \right\} \right]^2}.$$

Interestingly, the limits for this derivative are identical in the left and right tail. The latter is given by $\lim_{u_1 \rightarrow 1} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha; \sigma_1, \sigma_2, b) = b \cdot \sigma_2 / \sigma_1$ which is bounded and non-zero for $b \neq 0$. Figure 3.3 uses $b = 1/2$, $\sigma_1 = 1$ and $\sigma_2 = 3$.

For ease of notation denote by $t_\nu(\cdot)$ and $t_\nu^{-1}(\cdot)$ the cdf and quantile function of the Student’s- t_ν distribution with ν degrees of freedom. Following Bernard and Czado (2015, Table 2), the C-quantile of the Student’s- t_ν copula having correlation parameter ρ and $\nu > 2$ is given by

$$C_{U_2|U_1=u_1}^{-1}(\alpha; \rho, \nu) = t_\nu \left[t_{\nu+1}^{-1}(\alpha) \sqrt{\frac{1-\rho^2}{1+\nu} \{ \nu + t_\nu^{-1}(u_1)^2 \}} + \rho t_\nu^{-1}(u_1) \right].$$

The derivative of $C_{U_2|U_1=u_1}^{-1}(\alpha; \rho, \nu)$ wrt u_1 has a less appealing formula given by

$$\begin{aligned} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha; \rho, \nu) &= t'_\nu \left[t_{\nu+1}^{-1}(\alpha) \sqrt{\frac{1-\rho^2}{1+\nu} \{ \nu + t_\nu^{-1}(u_1)^2 \}} + \rho t_\nu^{-1}(u_1) \right] \\ &\cdot \frac{\rho + t_{\nu+1}^{-1}(\alpha) \sqrt{\frac{1-\rho^2}{1+\nu} \frac{t_\nu^{-1}(u_1)^2}{\nu + t_\nu^{-1}(u_1)^2}}}{t'_\nu \{ t_\nu^{-1}(u_1) \}}, \end{aligned} \quad (22)$$

where $t'_\nu(\cdot)$ denotes the density of the Student’s- t_ν distribution. Figure 3.2 uses the parameters $\rho = 0$ and $\nu = 5$. For the more general case, where $\rho = 0$, $\nu > 2$ and $\alpha \in (0, 1) \setminus \{ \frac{1}{2} \}$, the limit of (22) is given by

$$\lim_{u_1 \rightarrow 1} \frac{\partial}{\partial u_1} C_{U_2|U_1=u_1}^{-1}(\alpha; 0, \nu) = \pm \sqrt{\left\{ \Gamma_{\nu(\alpha)}^{-1} \left(\frac{1+\nu}{2}, \frac{1}{2} \right) \right\}^{-1} - 1 \left\{ \frac{\Gamma_{\nu(\alpha)}^{-1} \left(\frac{1+\nu}{2}, \frac{1}{2} \right)}{1 - \Gamma_{\nu(\alpha)}^{-1} \left(\frac{1+\nu}{2}, \frac{1}{2} \right)} \right\}^{\frac{1+\nu}{2}}},$$

having the positive sign for $v(\alpha) = 2(1 - \alpha)$, $\alpha \in (1/2, 1)$, and the negative sign for $v(\alpha) = 2\alpha$, $\alpha \in (0, 1/2)$. Even though we do not present a general analytical form for the case $\rho \in (-1, 1)$, we would like to emphasize that numerical experiments have shown that the limit of (22) usually deviates from zero in case $\rho \neq 0$.

Proof of Proposition 2

Having the \sqrt{n} -consistent and asymptotically Gaussian estimator $\hat{\theta}$, the Delta-method yields

$$\sqrt{n} \left[\{\gamma(\hat{\theta}) - \mathbf{1}\} - \{\gamma(\theta_0) - \mathbf{1}\} \right] \rightarrow_{\mathcal{L}} \text{N} \left[0, \mathcal{J} \{\gamma(\theta_0)\} \Sigma(\theta_0) \mathcal{J} \{\gamma(\theta_0)\}^\top \right].$$

Using $\gamma_1(\hat{\theta}_1) > 1$, the distribution of $\hat{\mathcal{E}}$ can be approximated by

$$\begin{aligned} \text{P} \left(\hat{\mathcal{E}} \leq x \right) &= \text{P} \left\{ \frac{\gamma_2(\hat{\theta}_2) - 1}{\gamma_1(\hat{\theta}_1) - 1} \leq x \right\} \\ &= \text{P} \left[(1, -x) \{\gamma(\hat{\theta}) - \mathbf{1}\} \leq 0 \right] \\ &= \text{P} \left(\sqrt{n} \mathbf{1}(x)^\top \left[\{\gamma(\hat{\theta}) - \mathbf{1}\} - \{\gamma(\theta_0) - \mathbf{1}\} \right] \leq -\sqrt{n} \mathbf{1}(x)^\top \{\gamma(\theta_0) - \mathbf{1}\} \right) \\ &\approx \Phi \left(\frac{\sqrt{n} \mathbf{1}(x)^\top \{\mathbf{1} - \gamma(\theta_0)\}}{[\mathbf{1}(x)^\top \mathcal{J} \{\gamma(\theta_0)\} \Sigma(\theta_0) \mathcal{J} \{\gamma(\theta_0)\}^\top \mathbf{1}(x)]^{1/2}} \right) \quad \text{for large } n. \end{aligned}$$

□

Several explanatory variables revisited

Last but not least, let us revisit the situation when d risk factors X_1, \dots, X_d are involved in the analysis. The subsequent paragraph provides the technical foundation for the claim stated above that the asymptotic elasticity $\mathcal{E}_{k\ell}$ relying on two risk factors X_k and X_ℓ is equivalent to $\mathcal{E}_{k\ell}(x_{-k})$ relying on d risk factors X_1, \dots, X_d given that all components of x_{-k} converge simultaneously to infinity.

Formally, let the \mathbb{R}^d valued random variable $X = (X_1, \dots, X_d)^\top$ refer to d risk factors and define $X_{-k} = (X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_d)^\top$, i.e., random variable X_k is not included in X_{-k} . Denote by $F(x_1, \dots, x_d) = \text{P}(X_1 \leq x_1, \dots, X_d \leq x_d)$ the cdf of X and by $C(u_1, \dots, u_d)$ the

corresponding copula. The corresponding conditional cdf is denoted by

$$F_{X_k|X_{-k}=x_{-k}}(x_k) = \text{P}(X_k \leq x_k | X_{-k} = x_{-k}), \quad (23)$$

with $x_{-k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_d)^\top$. As $F_{X_k|X_{-k}=x_{-k}}(x_k)$ is strictly increasing in x_k , the conditional quantile is

$$Q_{X_k|X_{-k}=x_{-k}}(\alpha) = F_{X_k|X_{-k}=x_{-k}}^{-1}(\alpha) \quad \text{with } \alpha \in (0, 1). \quad (24)$$

Based on $U_j = F_j(X_j)$ and $u_j = F_j(x_j)$ with $U_j \sim \text{U}(0, 1)$ and $u_j \in (0, 1)$, $j = 1, \dots, d$, the conditional cdf (23) can be expressed in terms of the conditional copula by

$$C_{U_k|U_{-k}=u_{-k}}(u_k) = \text{P}(U_k \leq u_k | U_{-k} = u_{-k}),$$

where $\{U_{-k} = u_{-k}\} = \{F_{-k}(X_{-k}) = F_{-k}(x_{-k})\}$ refers to the event $\{F_1(X_1) = F_1(x_1), \dots, F_{k-1}(X_{k-1}) = F_{k-1}(x_{k-1}), F_{k+1}(X_{k+1}) = F_{k+1}(x_{k+1}), \dots, F_d(X_d) = F_d(x_d)\}$. Using the corresponding C-quantile $C_{U_k|U_{-k}=u_{-k}}^{-1}(\alpha)$ and unconditional quantiles, (24) can be rewritten as

$$Q_{X_k|X_{-k}=x_{-k}}(\alpha) = Q_k\{C_{U_k|U_{-k}=u_{-k}}^{-1}(\alpha)\}.$$

Using this C-quantile representation, the asymptotic elasticity is given by

$$\mathcal{E}_{k\ell} = \lim_{u_{-k} \rightarrow \mathbf{1}} \frac{Q_\ell(u_\ell) q_k\{C_{U_k|U_{-k}=u_{-k}}^{-1}(\alpha)\}}{q_\ell(u_\ell) Q_k\{C_{U_k|U_{-k}=u_{-k}}^{-1}(\alpha)\}} \frac{\partial}{\partial u_\ell} C_{U_k|U_{-k}=u_{-k}}^{-1}(\alpha), \quad \alpha \in (0, 1), \quad (25)$$

where $\lim_{u_{-k} \rightarrow \mathbf{1}}$ means that each component of u_{-k} converges to 1 and $\mathbf{1}$ is vector of ones consisting of $(d - 1)$ components. Since $C(1, \dots, 1, u_k, u_\ell, 1, \dots, 1) = D(u_k, u_\ell)$, where $D(u_k, u_\ell)$ is the copula between $U_k = F_k(X_k)$ and $U_\ell = F_\ell(X_\ell)$, (25) is - due to the continuity of

underlying functions - equivalent to

$$\mathcal{E}_{k\ell} = \lim_{u_\ell \rightarrow 1} \frac{Q_\ell(u_\ell)q_k\{D_{U_k|U_\ell=u_\ell}^{-1}(\alpha)\}}{q_\ell(u_\ell)Q_k\{D_{U_k|U_\ell=u_\ell}^{-1}(\alpha)\}} \frac{\partial}{\partial u_\ell} D_{U_k|U_\ell=u_\ell}^{-1}(\alpha), \quad (26)$$

where $D_{U_k|U_\ell=u_\ell}^{-1}(\alpha)$ refers to the C-quantile derived from the bivariate copula of U_k and U_ℓ . Note that the right hand side of (26) brings us back to the pleasant situation used for the proofs of Proposition 1, see Equation 11.

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