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The Diversification Delta: A Different Perspective ☆

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Abstract

Vermorken et al. (2012) introduce a new measure of diversification, the *Diversification Delta* based on the empirical entropy. The entropy as a measure of uncertainty has successfully been used in several frameworks and takes into account the uncertainty related to the entire statistical distribution and not just the first two moments of a distribution. However, the suggested Diversification Delta measure has a number of drawbacks that we highlight in this article. We also propose an alternative measure based on the exponential entropy which overcomes the identified shortcomings. We present the properties of this new measure and illustrate its usefulness in an empirical example of a portfolio of U.S. stocks and bonds.

Keywords: Portfolio Optimization, Diversification Measures, Risk Management
JEL: G11, G23, C58

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1. Introduction

Vermorken et al. (2012) introduce a new measure of diversification, which is based on higher moments of the return distribution of a portfolio. The proposed measure, called *Diversification Delta* (DD), is based on the concept of Shannon entropy, or information entropy, that can measure the uncertainty related to the entire statistical distribution and not just the first two moments of a distribution.

Investors typically diversify their portfolios with the aim of reducing their exposure to idiosyncratic risks of individual assets, while the correlation matrix of asset returns is regarded as the common metric for measuring portfolio diversification. However, the correlation matrix provides a quantification of the pairwise relation between two or more stochastic processes. As pointed out by Statman and Scheid (2008), the correlation matrix does not account for the impact of individual assets on the variance of the portfolio. Furthermore, modern portfolio theory quantifies the level of diversification by using the first two moments of the return distribution only.

As mentioned in Vermorken et al. (2012), different diversification measures have been proposed in the financial literature. Various researchers consider the use of the correlation matrix, as well as alternative measures such as clustering based methods, the portfolio diversification index, and the return gaps. For a detailed analysis of these methods, see, for example, Dopfel (2003), Brown and Goetzmann (2003), Rudin and Morgan (2006) and Statman and Scheid (2008). These models come as a response to the classic portfolio optimization model introduced by Markowitz (1952). Other models include indexes based on mean-variance analysis (MVA) such as the Sharpe ratio by Sharpe (1966) and indexes based on risk measures, such as the diversification index defined by Tasche (2006) which is based on the value at risk. Although the use of these indexes has been successful to a large extent, the diversification delta has the advantage of not being restricted to the first two moments of the return distribution. It also captures diversification by comparing return distributions of the assets before and after the portfolio is constructed. At the same time, it is relatively easy to calculate and interpret.

As argued by Vermorken et al. (2012), entropy captures the reduction in uncertainty as a portfolio of various assets becomes more diversified. Increased portfolio diversification reduces uncertainty and lowers entropy. The proposed DD measure has the advantage of straightforward application and interpretation for portfolios consisting of different asset classes. Based on an empirical example, using returns from different infrastructure indexes, the authors argue that the application of DD *[..] gives the portfolio manager a much clearer picture of the reality in the market than the correlation coefficient.* Therefore, DD presents an interesting contribution to the study of portfolio theory and diversification. However, the way the measure is constructed, based on the entropy of individual asset returns and the portfolio return distribution, leads to some issues that will be examined in this study. For example, it is easy to illustrate that the proposed measure does not provide a range of $0 \leq DD \leq 1$ as suggested by the authors. This is true, in particular, when assets with a different level of risk or variance are being combined, and makes appropriate interpretation of the measure quite difficult. Further, in some instances, the measure provides results that contradict what one would intuitively expect from a diversification measure.

The remainder of the paper is structured as follows. Section 2 analyzes the DD presented in Vermorken et al. (2012) and illustrates some shortcomings of the proposed measure. In Section 3, we present a revised measure that is also based on the entropy of a random variable but overcomes several of the problems of the original DD. Section 4 provides an empirical application of the original DD and the newly proposed measure. Conclusions and suggestions for future work are presented in Section 5.

2. The Diversification Delta

The diversification delta is based on the entropy as a measure of uncertainty. Originally, the entropy was defined in a statistical mechanics framework and it was introduced in information theory by Shannon (1948). Since then, entropy has successfully been used in measuring uncertainty in several areas such as applied mathematics, electrical engineering, computer science, physics and neuroscience among many others.

In econometrics, the discrete entropy is often maximized to fit probability functions, see, for example, Maasoumi (1993) and Ullah (1996). In the context of financial economics, the entropy and the conditional entropy have been used to define convex risk measures (Laeven and Stadje, 2010; Föllmer and Knispel, 2011), while entropy measures have been used to estimate distributions associated with financial data (Kitamura and Stutzer, 1997; Robertson et al., 2005).

Although the use of entropy has not become a widespread measure in the portfolio optimization literature or amongst practitioners, interesting studies can be found. In the seminal work of Philippatos and Wilson (1972), entropy is used for the first time in a portfolio optimization framework. Following this, Hua and Xingsi (2003) and Bera and Park (2008) reported that entropy can be an effective alternative to the MVA. More recently, entropy has received increased attention as it can capture heavy tails that are often present in financial return data (Urbanowicz et al., 2012; Dey and Juneja, 2012). Regarding the use of entropy to build diversification measures, Bera and Park (2008) and Meucci (2009) developed diversification measures based on maximizing the entropy.

The framework by which entropy is used in the definition of the diversification delta differs from the way it is used in other portfolio analysis methods. In this case, entropy plays the central role and is evaluated in both the assets and the portfolio with no further analysis needed. Also, as suggested by Campbell (1966), the exponential entropy is used to avoid singularities of the entropy while still letting the uncertainty speak for itself. The exponential entropy is also mentioned as a risk measure in Fabozzi (2012).

For a given portfolio P consisting of N assets $(X_1, ..., X_N)$ and weights $(w_1, ..., w_N)$, with $\sum_{i=1}^{N} w_i = 1$, Vermorken et al. (2012) define the diversification delta as

$$DD(P) = \frac{\exp\left(\sum_{i=1}^{N} w_i H(X_i)\right) - \exp\left(H\left(\sum_{i=1}^{N} w_i X_i\right)\right)}{\exp\left(\sum_{i=1}^{N} w_i H(X_i)\right)}, \tag{1}$$

where f is the density of X and $H(X) = -\int_x [f(x)] \log(f(x))dx$ is the differential entropy. The differential entropy is used as a measure of uncertainty. The estimator of the entropy they consider is the one developed in Stowell and Plumbley (2009). Given that we do not consider the discrete entropy, we refer to it simply as entropy. The higher the level of entropy, the higher the uncertainty and vice-versa. The DD is designed to measure the diversification effect of a portfolio by considering the entropy of the assets and comparing it with the entropy of the portfolio.

The diversification delta is defined as a ratio that compares the weighted individual assets and the portfolio. The use of such ratio constitutes an interesting idea which quantifies the effect of diversification. Also, the use of the entropy to measure uncertainty in this context is in itself an important contribution. However, the way the entropy is used to measure uncertainty entails a number of issues. First, the measure of uncertainty used in equation (1) must satisfy a number properties for the ratio to adequately measure portfolio diversification. Artzner et al. (1999), Rockafellar et al. (2006) and McNeil et al. (2005), among others, have thoroughly analyzed desirable properties for measures of risk and uncertainty in a portfolio optimization framework. In the case of the entropy, although it has proven to be very effective in measuring uncertainty, it is well known that it is not homogeneous, is not left-bounded and is not subadditive, see, for example, Cover and Thomas (1991). In the definition of the diversification delta, the exponential function is used to account for these issues, however, it is not achieved in an adequate framework, as we now discuss in further detail.

Recall that the diversification delta compares the individual assets with the portfolio using the measure of uncertainty. The first characteristic we notice is that, for the diversification delta to be positive, the measure of uncertainty must satisfy some kind of subadditivity that ensures that the numerator in (1) is positive. As noted by Artzner et al. (1999) and McNeil et al. (2005), subadditivity is a crucial feature for measures of uncertainty or risk as it reflects the fact that risk can be reduced by diversification. The authors also comment on the shortcomings of the use of measures that do not account for this, the most notable example being VaR (McNeil et al., 2005). Subadditivity is not satisfied in the definition of the diversification delta, which can in fact not only be negative, but is also not left-bounded. Vermorken et al. (2012) state that the DD should exhibit a range of outcomes between 0 and 1. The authors suggest that this is actually one of the great benefits of the measure allowing for straightforward interpretation. While a value of 0 represents no diversification, a *'[...] value of one indicates that only market risk remains in the portfolio and all idiosyncratic risk has been diversified.'* (page 68) Therefore, the measure is not meant to be negative, since this causes some difficulties with respect to its interpretation. As we will illustrate in an example, negative outcomes for DD are likely to occur even under very plausible scenarios. This is particularly worrying as it contravenes the essential idea of the diversification delta.

Homogeneity is another desirable property for measures of uncertainty, as it ensures that changes in the size of an asset or the portfolio are detected according to their magnitude[1]. Given that $H(aX) = H(X) + \log(|a|)$, see Cover and Thomas (1991), the left-hand side in the numerator of (1) is not homogeneous with respect to the asset. This means that changes in the size of the assets are not detected in the same way as changes in the portfolio, leading to inconsistencies in the measurement of diversification. In the following Examples 1 to 3, we highlight the shortcomings of the diversification delta we have described. For simplicity, we consider the bivariate case.

[1]A function F is homogenous if for a constant a, $F(ax) = aF(x)$.

In Example 1, we present a portfolio formed by two assets following a normal distribution. We consider the particular case when one of the assets is riskier than the other, indicated by a significantly higher variance. Such a scenario is realistic for real world portfolios, for example when combining defensive assets (e.g. bonds) with growth assets (e.g. equities). Unfortunately, Vermorken et al. (2012) in their analysis do not consider such a case, but rather focus on the behavior of the diversification delta when assets with identical or very similar variance are combined to create a portfolio.

Example 1. *Assume that the expected returns of Asset 1 are normally distributed with* $\mu_{X_1} = 0.05$ *and* $\sigma_{X_1} = 0.1$*, while expected returns of Asset 2 follow a normal distribution with* $\mu_{X_2} = 0.01$ *and* $\sigma_{X_2} = 0.02$*.*

Thus, the first asset yields a higher expected return with higher risk, measured by the standard deviation, while the second asset exhibits a lower expected return, but also a significantly lower standard deviation. The diversification delta for a portfolio with normally distributed assets can be expressed as (see Appendix A, equation (6)):

$$\begin{aligned} DD(P) &= \frac{\sigma_{X_1}^{w_1}\sigma_{X_2}^{w_2} - \sigma_P}{\sigma_{X_1}^{w_1}\sigma_{X_1}^{w_2}} \\ &= 1 - \frac{\sigma_P}{\sigma_{X_1}^{w_1}\sigma_{X_2}^{w_2}}. \end{aligned} \tag{2}$$

Let us first consider the case when asset returns are independent. For a portfolio consisting of Asset 1 and Asset 2, the diversification delta is then equal to:

$$DD(P) = 1 - \frac{1}{2}\sqrt{\frac{26}{25} \cdot \frac{5}{1}} \approx -0.14,$$

which is less than zero. Furthermore, it is clear that $DD(P)$ becomes more and more negative the smaller σ_{X_2} is. In fact, note that if $\sigma_{X_2} \to 0$, then $\sigma_P \to w_1\sigma_{X1} = 0.1w_1$ and, from equation (6), $DD(P) \to 1 - \frac{0.1w_1}{0.1 \cdot 0} = -\infty$.

This issue presents complications when interpreting the diversification delta. Figure (1) illustrates the results for the constructed portfolios using Asset 1 and Asset 2. On the left-hand side, similar to Vermorken et al. (2012), we consider an equal weighted portfolio and determine the coefficient of correlation ex ante from -1 to 1. On the right-hand side we consider a portfolio consisting of independent assets by changing the weight w_1 of Asset 1 from 0 to 1. The figure illustrates that the DD becomes negative once the coefficient of correlation is greater than -0.8. Note that we still observe that DD declines once the correlation coefficient increases, i.e. when the diversification is reduced. Therefore, in a relative way, the proposed measure still contains information on the benefits of diversification when the two assets are combined into a portfolio. However, the interpretation of DD becomes significantly more difficult in this case, since it can take on negative values. The right-hand side shows that the DD exhibits erratic behavior when the weights change, which further complicates its interpretation.

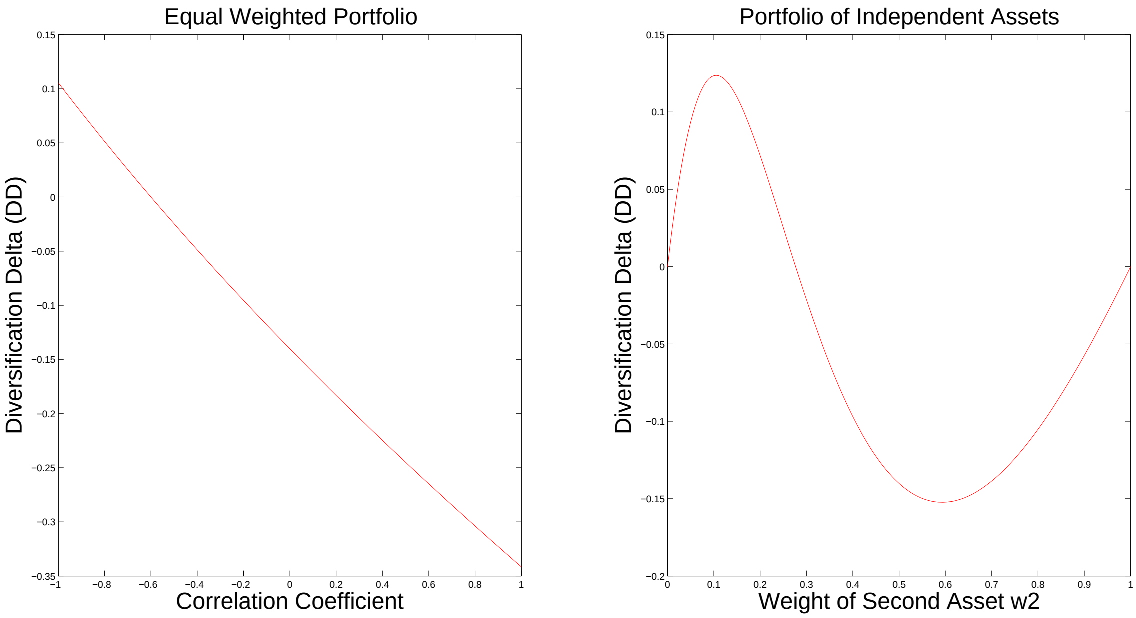

Figure 1: Diversification Delta as a function of the correlation coefficient and the portfolio weight w_2 for an exemplary two-asset portfolio. Expected returns of Asset 1 are normally distributed with $\mu_{X_1} = 0.05$ and $\sigma_{X_1} = 0.1$, while expected returns of Asset 2 follow a normal distribution with $\mu_{X_2} = 0.01$ and $\sigma_{X_2} = 0.02$. On the left-hand side we assume $w_1 = w_2 = 0.5$ and an ex ante determined correlation coefficient varying between -1 and 1. On the right-hand side we assume the assets are independent and an ex ante determined weight w_2 of the second asset.

Further, as mentioned earlier, the lack of homogeneity presents issues with the consistency of the diversification delta. It is a well-known fact that portfolios consisting of assets that are a linear combination of one asset offer no diversification. However, the diversification delta fails to detect this simple property and yields negative values for such portfolios. For simplicity, we will once more consider the bivariate case in our second example and construct a portfolio of assets which are a linear combination of each other.

Example 2. *Let P be a portfolio consisting of two assets, X and aX, with a being a positive constant. Using the property of the differential entropy, $H(aX) = H(X) + \log(|a|)$, and $w_1 +$*

$w_2 = 1$, *the diversification delta is:*

$$DD(P) = \frac{a^{w_2} - (1 + (a-1)w_2)}{a^{w_2}}.$$

This value does not depend on the entropy $H(X)$ and is only zero when one of the weights is zero and negative in any other case.

Consider for Example 2 the case where $a = 2$, that is, one of the assets is twice the other one. In this case the diversification delta is $DD(P) = \frac{2^{w_2} - (1+w_2)}{2^{w_2}}$. In Figure (2), we illustrate this case when the weight for Asset 2, w_2, is allowed to vary from $w_2 = 0$ up to $w_2 = 1$. It becomes obvious that for the illustrated example, the DD is negative for all cases except for either $w_2 = 0$ or $w_2 = 1$, where the DD takes on a value of zero. The measure reaches its minimum value for $w_2 = 0.4427$. However, while the constructed portfolio does not provide any diversification benefits, it is also neither less nor more diversified than the original assets 1 or 2. It is the lack of homogeneity in the left-hand side of the numerator of (1) which yields different results for different weights. For the specified example, an appropriate measure for diversification of a portfolio should be 0 for all constructed portfolios and should not depend on the choice of the portfolio weights.

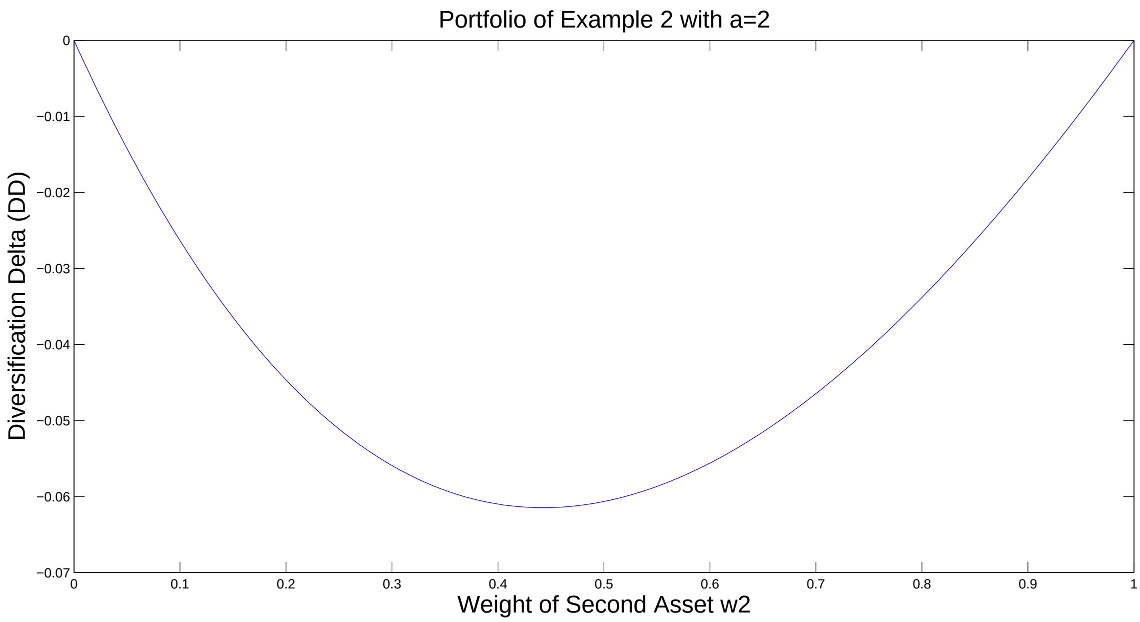

Figure 2: Diversification Delta as a function of portfolio weight w_2 for an exemplary two-asset portfolio with Asset 2 equal to 2*Asset 1. The constructed portfolio will not provide any diversification.

Let us finally consider Example 3, where we illustrate the lack of homogeneity of the original

DD leads to inconsistent results.

Example 3. *Investor 1 is building a portfolio consisting of assets* X_1 *and* X_2*. This investor determines that a portfolio with equal weights to be optimal, i.e.* $P_1 = \frac{1}{2}X_1 + \frac{1}{2}X_2$*. In a different market, investor 2 is building a portfolio from assets* $Y_1 = \frac{3}{2}X_1$ *and* $Y_2 = \frac{3}{4}X_2$*.*[2] *This investor determines optimal weights of* $w_1 = \frac{1}{3}$ *and* $w_2 = \frac{2}{3}$*, yielding the same portfolio* $P_2 = \frac{1}{2}X_1 + \frac{1}{2}X_2$*. Given that both portfolios are the same and have the same underlying assets, one would expect that the two portfolios have the same diversification delta. However this is not the case (see Appendix B) such that*

$$DD(P_1) \neq DD(P_2).$$

This follows from the lack of homogeneity of the left-hand side of the numerator in equation (1).

Given the drawbacks we identified in the diversification delta, we now deal with the issue of defining an alternative measure.

3. A revised Diversification Delta (DD^*) measure

As we mentioned before, using a ratio that compares the uncertainty of individual assets with the uncertainty of the portfolio is an interesting approach to portfolio analysis. Also, given the ability of the entropy to measure uncertainty while taking into account higher moments, we agree with Vermorken et al. (2012) that a measure based on the entropy will provide a useful tool to quantify portfolio diversification. We now focus on the issue of defining a measure that overcomes the drawbacks of the original diversification delta, while still relying on the entropy to measure uncertainty. As we have seen, a measure of uncertainty should satisfy certain properties to be well defined. In particular, it is desirable that the measure is homogeneous and subadditive, see Artzner et al. (1999); McNeil et al. (2005). The measure should also be bounded between 0 and 1, while reflecting the level of portfolio diversification.

The differential entropy itself is not subadditive or homogeneous. Moreover, unlike the discrete entropy, it can be negative. The differential entropy is maximized by the normal distribution. That is, when X is a random variable with finite variance σ_X^2, we have:

$$H(X) \leq \log(\sqrt{2\pi e \sigma_X^2}), \tag{3}$$

[2]Note that the coefficients in these equations represent volume not weights.

see, for example, Cover and Thomas (1991).

An interesting case for analysis is the entropy of a constant, that is when $X = C$ and $\sigma_X = 0$. Although the density is not defined in this case, it is obvious that when a random variable tends to approach a constant, its differential entropy tends to approach $-\infty$. Although this is consistent with the idea that the smaller the entropy the less uncertainty there is, it makes entropy more difficult to deal with.

The issues we just discussed can be addressed by considering the exponential entropy as a measure of uncertainty. The exponential entropy satisfies the following properties for variables X and Y and constants $C \in \mathbb{R}$ and $\lambda > 0$,

(1) $\exp(H(X + C)) = \exp(H(X))$,

(2) $\exp(H(0)) = 0$ and $\exp(H(\lambda X)) = \lambda \exp(H(X))$,

(3) $\exp(H(X + Y)) \leq \exp(H(X)) + \exp(H(Y))$,

The proof of (1) and (2) can be found in Cover and Thomas (1991) and (3) is verified in Salazar (2014).

These properties imply that for random variables $X_1, ..., X_N$ and weights $(w_1, ..., w_N)$, with $\sum_{i=1}^{N} w_i = 1$:

$$\exp\left(H\left(\sum_{i=1}^{N} w_i\left(X_i\right)\right)\right) \leq \sum_{i=1}^{N} w_i \exp\left(H(X_i)\right).$$

This is straightforward for $N = 2$ and it follows inductively for higher values. Considering this and properties (1) to (3), we propose the following revised measure, the diversification delta DD^*:

$$DD^*(P) = \frac{\sum_{i=1}^{N} w_i \exp\left(H(X_i)\right) - \exp\left(H\left(\sum_{i=1}^{N} w_i X_i\right)\right)}{\sum_{i=1}^{N} w_i \exp\left(H(X_i)\right)}. \tag{4}$$

Note that the difference with the original diversification delta proposed by Vermorken et al. (2012) is that in the left-hand side of the numerator, we use the weighted geometric mean of the exponential entropies of the assets, instead of the weighted arithmetic mean. The estimator of the entropy we consider is also the one found in Stowell and Plumbley (2009). We now analyze Examples 1 to 3 using the new DD^* measure.

3.1. Examples 1 to 3 revisited.

Considering Example 1, in the bivariate gaussian case, the new DD^* is equal to[3]

$$
\begin{aligned}
DD^*(P) &= \frac{(w_1\sigma_{X_1} + w_2\sigma_{X_2}) - \sigma_P}{(w_1\sigma_{X_1} + w_2\sigma_{X_2})} \\
&= 1 - \frac{\sigma_P}{(w_1\sigma_{X_1} + w_2\sigma_{X_2})},
\end{aligned} \tag{5}
$$

with $\sigma_P^2 = w_1^2\sigma_{X_1}^2 + 2\rho w_1 w_2 \sigma_{X_1}\sigma_{X_2} + w_2^2\sigma_{X_2}^2$. Note that, when the variances are the same, this measure and the original diversification delta of equation (6) are equal to $1 - \frac{\sigma_P}{\sigma_{X_1}}$. In equations (6) and (7), we see the differences between the measures exemplified. It shows why investing in one asset with low variance affects the measures in a very different way. In Figure 3 we replicate Figure 1 using the new diversification measure DD^*. In this figure we see that all values are between 0 and 1 and exhibit a less erratic behavior in comparison to the original diversification delta.

Now, considering Example 2, let P be a portfolio consisting of assets $(X_1, ..., X_N)$ which are all positive linear combinations of an asset X. Hence, $X_i = a_i X + b_i$, for constants $a_i > 0$ and b_i with $i \in \{1, ..., N\}$. Using the properties of the differential entropy it is easy to show (see Appendix C) that

$$
\sum_{i=1}^{N} w_i \exp\left(H(X_i)\right) = \exp\left(H\left(\sum_{i=1}^{N} w_i X_i\right)\right),
$$

such that the revised measure of diversification takes on a value of zero in this case. Therefore, the revised measure will be equal to zero for this case, where no diversification is achieved, since only linear combinations of an individual asset x are being combined.

Finally, for Example 3 consider the same notations as previously, in particular $c_1 = \exp(H(X_1))$, $c_2 = \exp(H(X_2))$ and $c_3 = \exp(H(\frac{1}{2}X_1 + \frac{1}{2}X_2))$. It is easy to show (see Appendix D) that for the new measure, we get

$$
DD^*(P_1) = DD^*(P_2) = \frac{c_1 + c_2 - 2c_3}{c_1 + c_2}.
$$

Given the homogeneity of the exponential entropy, the same holds for similar constructions. From the analysis, the DD^* measure is always between 0 and 1, therefore, it is equal to 1 when

[3] See Appendix A, equation (7).

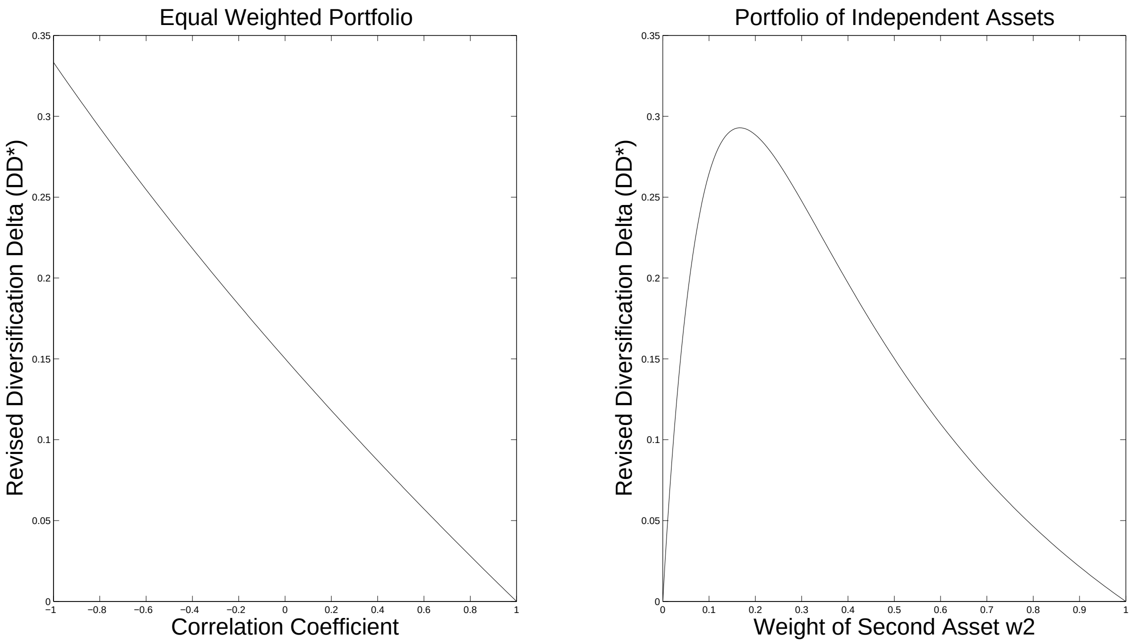

Figure 3: New Diversification Delta DD^* as a function of coefficient of correlation and portfolio weight w_2 for an exemplary two-asset portfolio. Expected returns of Asset 1 are normally distributed with $\mu_{X_1} = 0.05$ $\sigma_{X_1} = 0.1$, while expected returns of Asset 2 follow a normal distribution with $\mu_{X_1} = 0.01$ $\sigma_{X_1} = 0.02$. On the left-hand side we assume $w_1 = w_2 = 0.5$ and an ex ante determined correlation coefficient varying between -1 and 1. On the right-hand side we assume the assets are independent and an ex ante determined weight of the second asset.

the portfolio is constant (no risk) and 0 when the assets are a positive linear combination of a single asset (perfect positive dependence). Also, given the homogeneity of the exponential entropy, changes in the size of the portfolio and the assets are detected according to their magnitude.

3.2. Diversification Delta and the Sharpe Ratio

Let us now consider how the original DD and the newly derived Diversification Delta measure DD^* relate to the Sharpe ratio that measures reward-to-variability. Assume that the expected returns of Asset 1 are normally distributed with $\mu_1 = 0.05$ and $\sigma_1 = 0.1$, while expected returns of Asset B follow a normal distribution with $\mu_2 = 0.01$ and $\sigma_2 = 0.02$. So the first asset yields a higher expected return with higher risk, measured by the standard deviation, while the second asset has a lower expected return, but also a significantly lower standard deviation. Assume that the coefficient of correlation equals $\rho = 0.3$ for this example and that the portfolio only consists of these two assets, i.e. $w_1 + w_2 = 1$. We now calculate values of the Sharpe ratio, the original DD and the revised DD^* for portfolios where w_1 and w_2 are determined ex ante. In other words, we express the considered diversification, risk and reward-to-variability measures as a function of w_1, the weight for the more risky asset.

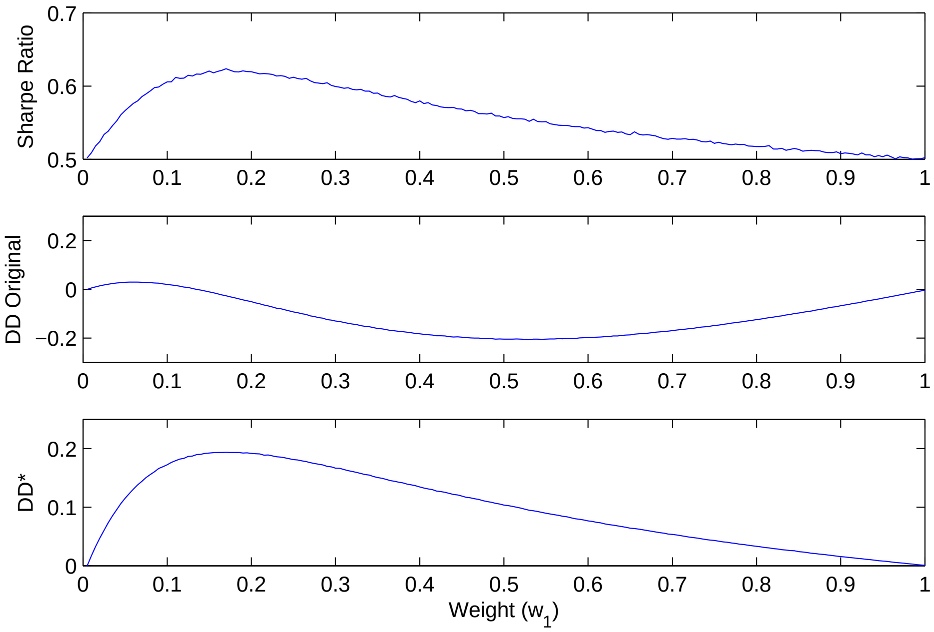

Figure 4: Sharpe Ratio *(upper panel)*, original Diversification Delta (DD) *(middle panel)* and revised Diversification Delta DD^* *(lower panel)* as a function of the coefficient of portfolio weight w_1 for the exemplary two-asset portfolio. Expected returns r_1 of Asset 1 are normally distributed with $\mu_1 = 0.05$ and $\sigma_1 = 0.1$. Asset 1 is combined with another Asset 2 with weight w_2 ($w_1 + w_2 = 1$). Returns of Asset 2 are normally distributed returns $\mu_2 = 0.01$ and $\sigma_2 = 0.02$. The coefficient of correlation between returns from Asset 1 and Asset 2 is set to be 0.3. Both the DD^* and the Sharpe Ratio are maximized for w_1=0.12

Figure 4 once more illustrates that the original DD does not really contain any information that can be easily interpreted by an investor. The measure takes on negative values for $0.15 \leq w_1 \leq 1$ which makes it almost impossible to create meaningful results from this measure. On the other hand, we observe that for normally distributed returns, the Sharpe ratio and the revised Diversification Delta DD^* are highly correlated. This makes sense, since the entropy of the normal distribution is entirely defined by its variance, therefore, a combination of assets with the highest Sharpe ratio will also yield a high DD^*. Both measures are initially increasing, take on their maximum value for $w_1 = 0.12$ and then decrease. Since, both individual assets provide a Sharpe ratio of 0.5, this is the lowest possible outcome for the ratio when either $w_1 = 1$ or $w_1 = 0$. Overall, we find that for the different portfolios we obtain $0.5 \leq SR \leq 0.6238$.

The revised DD^*, on the other hand, takes on a value of 0 when the entire investment is either allocated to Asset 1 or Asset 2 (no diversification) and increases when the two assets are combined. It reaches a maximum value for $w_1 = 0.12$ where we obtain $DD^* = 0.1936$. Overall, we conclude that when returns follow a normal distribution, DD^* and the SR yield very similar results with respect to the portfolio that maximizes these measures. Note, however, that unlike the SR, the DD^* considers the variance of the different assets (and not only of the portfolio) and is not dependent on the expected return. In that sense, the DD^* measures whether the risk is diversified away by the portfolio and is only dependent on risk. It must be emphasized that real-world asset returns are not normally distributed but will exhibit skewness, excess kurtosis and other features that make the empirical return distribution deviate from a normal distribution. Therefore, under real-world scenarios, the SR and DD^*-optimal portfolios may be very different. We will now investigate the behavior of DD^*, using empirical data from U.S. equity, bond and Treasury bill returns.

4. An Empirical Example

In this section we investigate the behavior of the revised measure DD^* in an empirical example. The estimator of the entropy we consider is the one developed in Stowell and Plumbley (2009). We examine stylized portfolios containing both growth assets such as equities and more defensive assets such as bonds. A combination of such assets is particularly interesting for pension funds. There are a myriad of asset allocation approaches currently implemented in approved pension funds portfolios. From defensive strategies to growth strategies. In this study, we consider one of the most popular portfolios in pension funds analysis, namely the 60/40 default portfolio with an asset allocation of 60% equity and 40% in bonds. These empirical asset returns exhibit very different levels of volatility, a situation where the original Diversification Delta will fail to provide meaningful results. Instead, we employ the revised Diversification Delta DD^* in comparison to alternative portfolio risk or performance measures.

4.1. Static Case

We employ U.S. asset class monthly returns for the period January 1970 to December 2013. The MSCI U.S. Equity Index is used as the proxy for broad equity returns. The proxy for U.S. bonds is a spliced time series of three data sources, namely, returns derived from the Robert J. Shiller bond yields from January 1970 to December 1972, the Lehman Brothers U.S. Government Long Term Bond Index from January 1973 to December 1998, and the Citigroup

Table 1: Descriptive statistics and performance metrics of the asset classes and investment portfolios from January 1970 to December 2013 (consisting of 528 monthly return observations). The asset ‘Equity’ denotes the MSCI U.S. Equity Index. ‘Bonds’ denotes the spliced time series of bond returns as a proxy for U.S. defensive assets. The U.S. Government Treasury Bills denote the spliced time series of bank bill returns as a proxy for the U.S. risk-free asset. ‘60/40’ denotes the 60/40 target risk fund portfolio with a constant asset allocation of 60% equity and 40% in bonds. The second to fifth column report the moments of the distribution of returns. The column ‘Sharpe’ denotes the monthly Sharpe ratio. The column ‘Max Draw’ denotes the maximum drawdown which is the largest percentage loss in the value of the asset or portfolio from its highest historical peak. The column ‘VaR’ refers to the historical 95% value-at-risk (VaR). The column ‘ETL’ refers to the historical 95% expected tail loss (ETL).

Asset	Mean	Std	Skewness	Kurtosis	Sharpe	Max Draw	VaR	ETL
Equity	0.0094	0.0462	-0.5352	4.8404	0.1119	-0.5039	-0.0728	-0.1015
Bonds	0.0062	0.0197	0.2110	8.8510	0.1031	-0.1733	-0.0225	-0.0375
Risk Free Rate	0.0042	0.0027	0.5354	3.5988	n/a	0.0000	0.0000	0.0000
60/40	0.0081	0.0307	-0.3215	4.2853	0.1276	-0.3205	-0.0432	-0.0642

U.S. Broad Investment Grade Bond Index from January 1999 to December 2013. The U.S. Government 1 month Treasury bills are employed as the proxy for the risk-free asset which is sourced from Kenneth French’s website. Table 1 reports the descriptive statistics for these three broad asset classes as well as the 60/40 DOA target risk fund portfolio with investments of 60% in equities and 40% in bonds.

Figure 5 provides a plot of three measures for the entire time period, namely the Sharpe Ratio, the standard deviation and the revised DD^* for a portfolio consisting of investments in equity and bonds. Similar to the exercise in Figure 4, we calculate the values of these measures for portfolios with different allocations to the equity index (w_1), ranging from $w_1 = 0$ up to $w_1 = 1$. In other words, we report results for the Sharpe Ratio, the standard deviation and the DD^* for portfolios with allocation to equities between 0% and 100%. The coefficient of correlation between the empirical returns from the two asset classes is estimated to be 0.2531. Clearly, empirical data on stock and bond returns is not normally distributed as illustrated in Table 1. We find that for empirical returns the measures provide quite different results for the portfolio. When considering asset returns for the entire 44 year period, we estimate $w_1 = 0.08$ in equities and $w_2 = 0.92$ in bonds yield the minimum variance portfolio when combining the

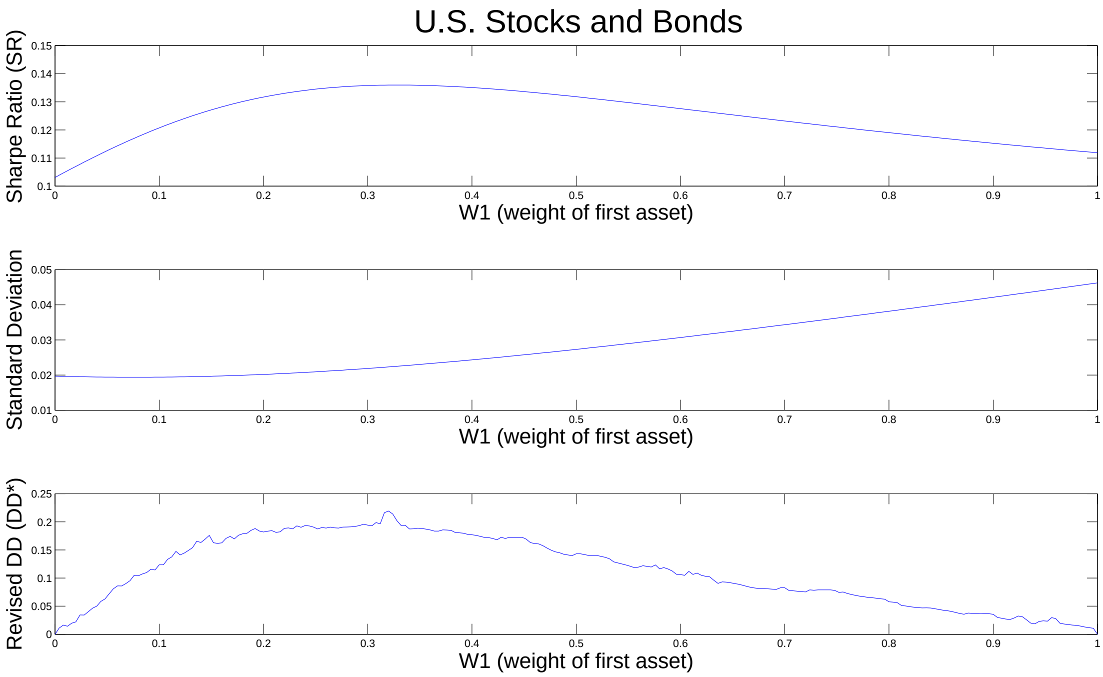

Figure 5: Sharpe Ratio *(upper panel)*, Portfolio standard deviation *(middle panel)* and revised Diversification Delta DD^* *(lower panel)* for different allocations to equities (w_1), ranging from $w_1 = 0$ up to $w_1 = 1$. The portfolio consists of two asset classes only: U.S. equity and U.S. bonds. We consider monthly returns for the time period January 1970 to December 2013, the coefficient of correlation between the empirical returns from the two asset classes is estimated to be 0.2531. The Sharpe Ratio is maximized for $w_1 = 0.33$, the Standard Deviation is minimized for $w_1 = 0.02$ and the Revised Diversification Delta DD^* is maximized $w_1 = 0.32$.

two asset classes. The portfolio that yields the maximum Sharpe ratio exhibits weights of $w_1 = 0.33$ in equity and $w_2 = 0.67$ in bonds. On the other hand, the combination of weights yielding the maximum Diversification Delta $DD^* = 0.22$ has weights of $w_1 = 0.32$ in equity and $w_2 = 0.68$ in bonds. Generally, for empirical data on asset returns, the diversification delta, taking into account also higher moments of the distribution yields a different 'optimal' portfolio than the Sharpe ratio, that is based on the mean and the variance only. Interestingly, for the considered measures, the DD^*-optimal portfolio and the SR-optimal portfolio allocate approximately one third of the portfolio to equities, when the entire 44 year period of returns is considered.

4.2. Rolling Window Analysis

In the following section we will investigate how the 60/40 portfolio compares to portfolios that are optimal with respect to return-to-variability, diversification or risk in a dynamic setting. Thus, we will compare the 60/40 portfolio to portfolios that are optimal with respect to maximizing the Sharpe ratio, maximizing the Diversification Delta DD^* or constructing the minimum variance portfolio (MVP). Figure 6 provides results for a rolling window analysis of the 60/40 portfolio versus optimal portfolios created according to the Diversification Delta DD^*, the Sharpe Ratio and the Minimum Variance criterion. Rolling windows are based on a five year period (i.e. 60 observations), such that the first window contains observations for the period January 1970 - December 1974, while the last window contains observations for the period January 2009 - December 2013. At each monthly time step, optimal portfolios are constructed according to the considered criteria, i.e. we calculate the corresponding weights for equity and bonds for the portfolio yielding the maximum DD^*, the maximum Sharpe ratio and the minimum variance. Figure 6 provides a plot of the calculated weights for the investments in the equity index for the optimal portfolios for each of the criteria versus the 60/40 portfolio. Naturally, for the latter, in each time step, the weights are constant whereby 60% of the portfolio is allocated to equities and 40% to bonds.

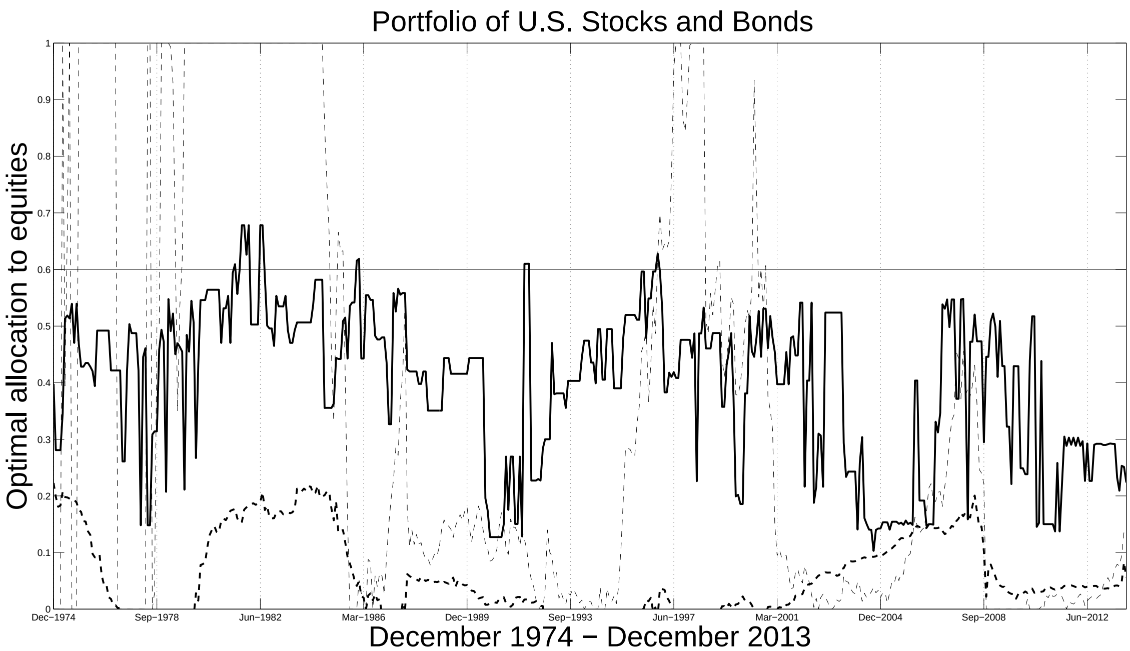

Figure 6: Rolling window weights for optimal asset allocation to equities with respect to various criteria: optimal weights for the maximum Diversification Delta DD^* (bold), maximum Sharpe Ratio (dashed), Minimum Variance Portfolio (bold dashed) versus the default 60/40 portfolio (solid). Rolling windows are based on a five year period (60 observations), such that the first window contains observations for the period January 1970 - December 1974, while the last window contains observations for the period January 2009 - December 2013.

We find that there are significant differences in the asset allocation weights for optimal portfolios according to the three criteria. Sharpe ratio-maximizing portfolios exhibit the greatest variability of portfolio weights through time, particularly in the first five years. Interestingly, it usually allocates a high proportion of investments in U.S. bonds, in particular during the periods from 1985 to 1994 and towards the end of the sample period. Over the entire period, the average asset allocation to equities to create the Sharpe ratio optimal portfolio is 34%. Thus, while in general the 60/40 portfolio has a very different allocation to equity and bonds than the Sharpe ratio optimal portfolio, the average allocation for our rolling window analysis is not too far from an allocation of 60% in equities.

For the minimum variance portfolio, as expected, we find a high asset allocation in bonds over the full sample period. For some periods the minimum variance portfolio exhibits a 100% allocation in bonds. Between 1979 and 1987 as well as for the end of the sample period, we find that the minimum variance portfolio yields a higher allocation to equity assets, however, the optimal weight in equity investments never exceeds 22%. For the entire period, the average asset allocation in equities in the MVPs is 5%.

Finally, we examine the behavior of the revised Diversification Delta DD^* through time. Recall that unlike the Sharpe ratio and the minimum variance portfolio, DD^* also takes into account higher moments of the return distribution for the individual assets and the portfolio. Interestingly, in comparison to constructed Sharpe ratio optimal portfolios, DD^*-optimal portfolios seem to exhibit a clearly lower variation in portfolio weights. The average allocation to equity is approximately 40%.

4.3. Performance Analysis

The final part of this paper investigates the performance of different strategies over the 44 year time horizon. Thus, we examine the performance of strategies that involve constructing Sharpe ratio optimal portfolios, DD^*-optimal portfolios or the MVP in each monthly time step against the performance of the 60/40 target risk fund portfolio.

To create the portfolio, in each monthly time step we consider a five year period (60 months) of historical returns for the assets and then select the optimal portfolio weights such that they (i) maximize the SR, (ii) maximize DD^* or, (iii) yield the MVP, or (iv) are always 60% equity and 40% in bonds. Then we calculate the return of each strategy for the next month before the portfolios are restructured. We ignore transaction costs for restructuring the portfolios at each time step.

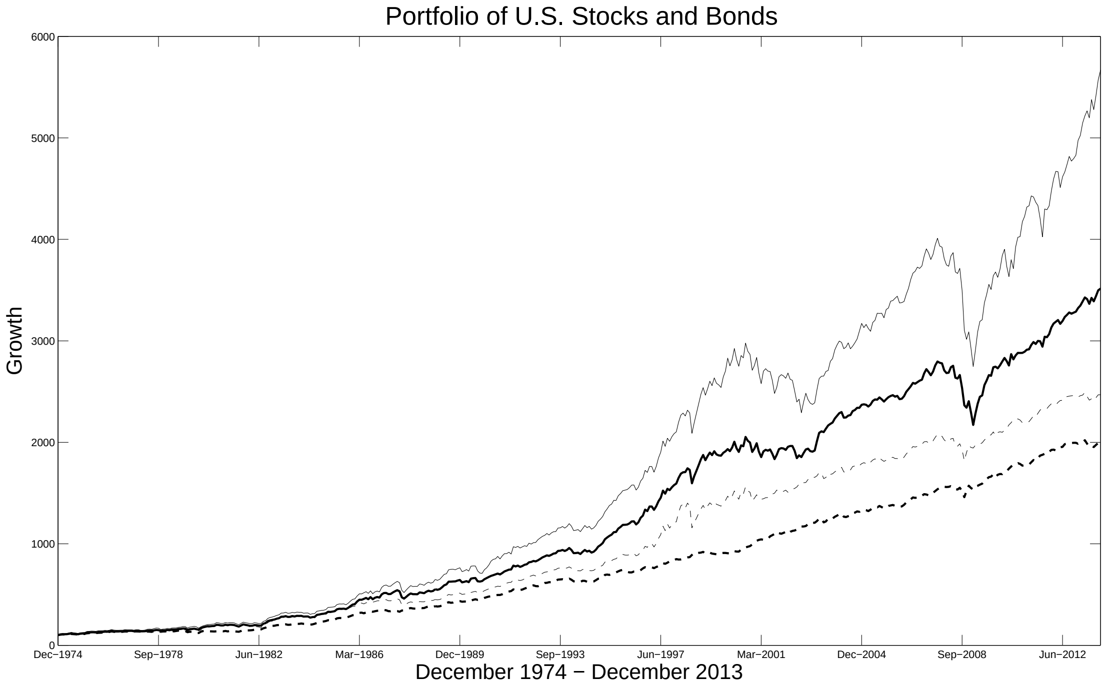

Figure 7: Performance of the portfolios using optimal asset allocation to equities with respect to different criteria: optimal weights for maximum Diversification Delta DD^* (bold), maximum Sharpe Ratio (dashed), Minimum Variance Portfolio (bold dashed) versus the default 60/40 portfolio (solid). Rolling windows are based on a five year period (60 monthly observations), such that the first window contains observations for the period January 1970 - December 1974, while the last window contains observations for the period January 2009 - December 2013. Portfolios are reconstructed every month.

Figure 7 illustrates a plot for the growth in portfolios based on the different strategies. As could be expected the MVP strategy yields the lowest overall return among all strategies, however, it is the least volatile. The strategy that uses SR optimal portfolios seems to perform quite well from the beginning but exhibits a significant drop afterwards. On the other hand, the DD^* optimal strategy exhibits a less severe decrease in the value of its portfolio during the 2008 crisis than the 60/40 portfolio and creates an overall much higher return than the SR optimal and MVP optimal strategies. The DD^* optimal strategy provides an overall higher return than the SR optimal strategy, but also yields a lower standard deviation of returns. One could argue that according to these criteria, portfolios that are created with respect to maximizing DD^*, outperform SR optimal portfolios. Interestingly, none of the strategies are able to outperform the 60/40 target risk fund portfolio in terms of the overall growth of the portfolio.

Table 2: Descriptive statistics and performance metrics of the various investment strategies across the out-of-sample performance period from January 1975 to December 2013 (consisting of 468 monthly return observations). All strategies employ the previous 60 months of returns to construct the following portfolios: optimal Diversification Delta (DD*), optimal Sharpe Ratio, minimum variance portfolio, 60/40 target risk portfolio with an asset allocation of 60% equities and 40% bonds, 100% equities and 100% bonds. The second to fifth column report the moments of the distribution of returns. The column 'Sharpe' denotes the monthly Sharpe ratio. The column 'Max Draw' denotes the maximum drawdown which is the largest percentage loss in the value of the asset or portfolio from its highest historical peak. The column 'VaR' refers to the historical 95% value-at-risk (VaR). The column 'ETL' refers to the historical 95% expected tail loss (ETL).

Strategy	Mean	Std	Skewness	Kurtosis	Sharpe	Max Draw	95% VaR	95% ETL
DD^*	0.0079	0.0238	-0.2581	5.4359	0.1591	-0.2236	-0.0305	-0.0476
Sharpe Ratio	0.0072	0.0269	-0.1247	9.5278	0.1153	-0.1739	-0.0297	-0.0545
MVP	0.0065	0.0167	0.4065	9.0341	0.1515	-0.1505	-0.0165	-0.0291
60/40 DOA	0.0091	0.0293	-0.4322	4.4852	0.1702	-0.3151	-0.0395	-0.0598
Equity	0.0109	0.0453	-0.6506	5.1599	0.1497	-0.5039	-0.0683	-0.0991
Bonds	0.0064	0.0165	0.3862	9.3061	0.1395	-0.1505	-0.0175	-0.0288

However, as indicated by Table 2, the returns of the 60/40 portfolio are more volatile than those ofthe DD^*-optimal strategy. On the other hand, the 60/40 portfolio also achieves higher mean returns but higher standard deviation of returns than the DD^*, the MVP and the SR optimal strategies. This emphasizes the risky nature of the 60/40 portfolio against other alternatives. The highest returns (but also highest risk measured by the variance of the created returns) would have been achieved by a fund that invests 100% in equity, while the lowest average returns (coinciding with the lowest risk) would have been achieved for an investment of 100% in bonds.

Table 2 reports the implementation of the various investment strategies from January 1975 to December 2013. The DD^* portfolio yields return and standard deviation metrics which are less risky than 100% equities but riskier than 100% bonds. We can also observe that the DD* portfolio exhibits a higher Sharpe Ratio in the out-of-sample period than the Sharpe optimal portfolio. The DD^* portfolio reports the second highest Sharpe ratio and is only marginally lower than the 60/40 portfolio. Furthermore, the DD* portfolio exhibits the lowest maximum drawdown statistics of all the investment strategies. Finally, the DD^* reports the third lowest 95% VaR and 95% ETL statistics behind the MVP and U.S. bonds. The key findings from Table 2 suggest that the DD* portfolio offers desirable portfolio characteristics to an investor without the sudden changes in asset allocation associated with classic portfolio optimization.

Overall, our results indicate that the revised Diversification Delta DD^* offers an alternative criterion that can be used to construct portfolios. For our simple empirical example, the DD^* portfolio yields different allocations to equity and bonds in comparison to SR optimal portfolios or MVPs. Portfolios that are constructed based on optimizing DD^* perform quite well and the application of the Diversification Delta should be investigated in future applications to portfolio management.

5. Conclusions

Vermorken et al. (2012) introduce a new measure of diversification, the *Diversification Delta* based on the empirical entropy of financial returns for individual assets or a portfolio. The entropy as a measure of uncertainty has successfully been used in several frameworks and takes into account the uncertainty related to the entire statistical distribution and not just the first two moments of a distribution. We illustrate that the suggested Diversification Delta measure has a number of drawbacks in particular when risky assets such as equities are combined with asset classes which exhibit a lower risk profile. We also propose a revised measure that is based on the exponential entropy which overcomes some of the identified shortcomings of the original diversification delta metric.

We present and demonstrate the properties of this new measure and illustrate the usefulness of the revised Diversification Delta (DD^*) in an application to a portfolio of U.S. stocks and bonds. Our findings suggest that the revised Diversification Delta offers an alternative criterion that can be used to construct optimally diversified portfolios. Portfolios that are optimal with respect to maximizing the Diversification Delta yield very different allocations to equity and bonds in comparison to portfolios that are constructed by optimizing the Sharpe Ratio or creating Minimum Variance portfolios. In an out-of-sample analysis, Diversification Delta optimal portfolios outperform Sharpe Ratio-optimal portfolios by creating higher average returns with lower standard deviation. Based on this evidence, we recommend the application of the revised Diversification Delta metric in future applications to portfolio management.

Interestingly, none of the created strategies are able to outperform the 60/40 portfolio in terms of the overall growth rate over a 39 year time horizon. This also illustrates that a simple asset allocation of 60% equity and 40% in bonds as it is the case for several pension funds performs quite well when a longer time horizon for the investment is assumed.

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Appendices

A. Diversification Delta (DD) and revised Diversification Delta (DD^*) for a bivariate portfolio normally distributed assets

Let $P = (w_1X_1 + w_2X_2)$ be a portfolio where the two assets are normal, $X_1 \sim N(0, \sigma^2_{X_1})$ and $X_2 \sim N(0, \sigma^2_{X_2})$ and the weights are positive and satisfy $w_1 + w_2 = 1$. In this case $P \sim N(0, \sigma^2_P)$, with $\sigma^2_P = w_1^2\sigma^2_{X_1} + 2\rho w_1 w_2 \sigma_{X_1}\sigma_{X_2} + w_2^2\sigma^2_{X_2}$, where ρ is the correlation between the assets.

Note that the entropy of a normally distributed variable X, with variance σ^2_X, is $\log(\sqrt{2\pi e \sigma^2_X})$ (see Cover and Thomas (1991)). Therefore,, considering equations (1) and (3), the Diversification Delta (DD) and the revised Diversification Delta (DD^*) in this bivariate normal case are

$$
\begin{aligned}
DD(P) &= \frac{\exp\left(w_1 \log\left(\sqrt{2\pi e \sigma^2_{X_1}}\right) + w_2 \log\left(\sqrt{2\pi e \sigma^2_{X_2}}\right)\right) - \exp\left(\log\left(\sqrt{2\pi e \sigma^2_P}\right)\right)}{\exp\left(w_1 \log\left(\sqrt{2\pi e \sigma^2_{X_1}}\right) + w_2 \log\left(\sqrt{2\pi e \sigma^2_{X_2}}\right)\right)} \\
&= \frac{\left[\exp\left(\log\left(\sqrt{2\pi e}\sigma_{X_1}\right)\right)\right]^{w_1} \cdot \left[\exp\left(\log\left(\sqrt{2\pi e}\sigma_{X_2}\right)\right)\right]^{w_2} - \sqrt{2\pi e}\sigma_P}{\left[\exp\left(\log\left(\sqrt{2\pi e}\sigma_{X_1}\right)\right)\right]^{w_1} \cdot \left[\exp\left(\log\left(\sqrt{2\pi e}\sigma_{X_2}\right)\right)\right]^{w_2}} \\
&= \frac{\left(\sqrt{2\pi e}\sigma_{X_1}\right)^{w_1} \cdot \left(\sqrt{2\pi e}\sigma_{X_2}\right)^{w_2} - \sqrt{2\pi e}\sigma_P}{\left(\sqrt{2\pi e}\sigma_{X_1}\right)^{w_1} \cdot \left(\sqrt{2\pi e}\sigma_{X_2}\right)^{w_2}} \\
&= \frac{\left(\sqrt{2\pi e}\right)^{w_1+w_2} \cdot \sigma^{w_1}_{X_1}\sigma^{w_2}_{X_2} - \sqrt{2\pi e}\sigma_P}{\left(\sqrt{2\pi e}\right)^{w_1+w_2} \cdot \sigma^{w_1}_{X_1}\sigma^{w_2}_{X_2}} \\
&= \frac{\sigma^{w_1}_{X_1}\sigma^{w_2}_{X_2} - \sigma_P}{\sigma^{w_1}_{X_1}\sigma^{w_2}_{X_1}} \\
&= 1 - \frac{\sigma_P}{\sigma^{w_1}_{X_1}\sigma^{w_2}_{X_2}},
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
DD^*(P) &= \frac{w_1 \exp\left(\log\left(\sqrt{2\pi e \sigma^2_{X_1}}\right)\right) + w_2 \exp\left(\log\left(\sqrt{2\pi e \sigma^2_{X_2}}\right)\right) - \exp\left(\log\left(\sqrt{2\pi e \sigma^2_P}\right)\right)}{w_1 \exp\left(\log\left(\sqrt{2\pi e \sigma^2_{X_1}}\right)\right) + w_2 \exp\left(\log\left(\sqrt{2\pi e \sigma^2_{X_2}}\right)\right)} \\
&= \frac{w_1\sqrt{2\pi e}\sigma_{X_1} + w_2\sqrt{2\pi e}\sigma_{X_2} - \sqrt{2\pi e}\sigma_P}{w_1\sqrt{2\pi e}\sigma_{X_1} + w_2\sqrt{2\pi e}\sigma_{X_2}} \\
&= \frac{(w_1\sigma_{X_1} + w_2\sigma_{X_2}) - \sigma_P}{(w_1\sigma_{X_1} + w_2\sigma_{X_2})} \\
&= 1 - \frac{\sigma_P}{(w_1\sigma_{X_1} + w_2\sigma_{X_2})},
\end{aligned}
\tag{7}
$$

B. Original Diversification Delta for Example 3

Investor 1 is building a portfolio from assets X_1 and X_2. This investor determines that a portfolio with equal weights to be optimal, i.e. $P_1 = \frac{1}{2}X_1 + \frac{1}{2}X_2$. In a different market, investor 2 is building a portfolio from assets $Y_1 = \frac{3}{2}X_1$ and $Y_2 = \frac{3}{4}X_2$.[4] This investor determines optimal weights of $w_1 = \frac{1}{3}$ and $w_2 = \frac{2}{3}$, yielding the same portfolio $P_2 = \frac{1}{2}X_1 + \frac{1}{2}X_2$.

Let $c_1 = \exp(H(X_1))$, $c_2 = \exp(H(X_2))$ and $c_3 = \exp(H(\frac{1}{2}X_1 + \frac{1}{2}X_2))$. Given that the portfolios are the same, from equation (1) in both cases we have

$$\begin{aligned} DD(P) &= \frac{\exp\left(w_1 H\left(Y_1\right) + w_2 H\left(Y_2\right)\right) - \exp(H(P))}{\exp(w_1 H(Y_1) + w_2 H(Y_2))} \\ &= \frac{\exp\left(w_1 H\left(Y_1\right) + w_2 H\left(Y_2\right)\right) - c_3}{\exp\left(w_1 H\left(Y_1\right) + w_2 H\left(Y_2\right)\right)}. \end{aligned}$$

In the case of Portfolio 1,

$$\begin{aligned} \exp\left(w_1 H\left(Y_1\right) + w_2 H\left(Y_2\right)\right) &= \exp\left(\frac{1}{2}\left(H(X_1) + H(X_2)\right)\right) \\ &= \left[\exp\left(\left(H(X_1) + H(X_2)\right)\right)\right]^{\frac{1}{2}} \\ &= \left(c_1 c_2\right)^{\frac{1}{2}}, \end{aligned}$$

and for Portfolio 2

$$\begin{aligned} \exp\left(w_1 H\left(Y_1\right) + w_2 H\left(Y_2\right)\right) &= \exp\left(\frac{1}{3}H\left(\frac{3}{2}X_1\right) + \frac{2}{3}H\left(\frac{3}{4}X_2\right)\right) \\ &= \left[\exp\left(H\left(\frac{3}{2}X_1\right)\right)\right]^{\frac{1}{3}} \cdot \left[\exp\left(H\left(\frac{3}{4}X_2\right)\right)\right]^{\frac{2}{3}} \\ &= \left[\frac{3}{2}\exp\left(H\left(X_1\right)\right)\right]^{\frac{1}{3}} \cdot \left[\frac{3}{2}\frac{\exp\left(H\left(X_2\right)\right)}{2}\right]^{\frac{2}{3}} \\ &= \left(\frac{3}{2}\right)^{\frac{1}{3}} c_1^{\frac{1}{3}} \cdot \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(\frac{c_2}{2}\right)^{\frac{2}{3}} \\ &= \left(\frac{3}{2}\right) \cdot \left(\frac{c_1 c_2^2}{4}\right)^{\frac{1}{3}}. \end{aligned}$$

Therefore, we get

$$DD(P_1) = \frac{(c_1 c_2)^{\frac{1}{2}} - c_3}{(c_1 c_2)^{\frac{1}{2}}} \neq DD(P_2) = \frac{\frac{3}{2}(\frac{c_1 c_2^2}{4})^{\frac{1}{3}} - c_3}{\frac{3}{2}(\frac{c_1 c_2^2}{4})^{\frac{1}{3}}}.$$

[4] Note that the coefficients in these equations represent volume not weights.

C. Revised Diversification Delta for combination of identical assets

Let P be a portfolio consisting of assets $(X_1, ..., X_N)$ which are all positive linear combinations of an asset X. Hence, $X_i = a_i X + b_i$, for constants $a_i > 0$ and b_i with $i \in \{1, ..., N\}$.

$$\begin{aligned}
\sum_{i=1}^{N} w_i \exp\left(H(X_i)\right) &= \sum_{i=1}^{N} w_i a_i \exp\left(H(X)\right) \\
&= \exp\left(H(X)\right) \sum_{i=1}^{N} w_i a_i \\
&= \exp\left(H\left(\sum_{i=1}^{N} w_i a_i X\right)\right) \\
&= \exp\left(H\left(\sum_{i=1}^{N} w_i a_i X + \sum_{i=1}^{N} w_i c_i\right)\right) \\
&= \exp\left(H\left(\sum_{i=1}^{N} w_i X_i\right)\right),
\end{aligned}$$

Therefore,

$$DD^*(P) = \frac{\sum_{i=1}^{N} w_i \exp\left(H(X_i)\right) - \exp\left(H\left(\sum_{i=1}^{N} w_i X_i\right)\right)}{\sum_{i=1}^{N} w_i \exp\left(H(X_i)\right)} = 0$$

D. Revised Diversification Delta for Example 3

Again, let $c_1 = \exp(H(X_1))$, $c_2 = \exp(H(X_2))$ and $c_3 = \exp(H(\frac{1}{2}X_1 + \frac{1}{2}X_2))$. Using (3), in both cases we have

$$\begin{aligned}
DD^*(P) &= \frac{w_1 \exp\left(H\left(Y_1\right)\right) + w_2 \exp\left(H\left(Y_2\right)\right) - \exp(H(P))}{w_1 \exp\left(H\left(Y_1\right)\right) + w_2 \exp\left(H\left(Y_2\right)\right)} \\
&= \frac{w_1 \exp\left(H\left(Y_1\right)\right) + w_2 \exp\left(H\left(Y_2\right)\right) - c_3}{w_1 \exp\left(H\left(Y_1\right)\right) + w_2 \exp\left(H\left(Y_2\right)\right)}.
\end{aligned}$$

For Portfolio 1,

$$\begin{aligned}
w_1 \exp\left(H\left(Y_1\right)\right) + w_2 \exp\left(H\left(Y_2\right)\right) &= \frac{1}{2}\left(\exp\left(H(X_1)\right) + \exp(H(X_2))\right) \\
&= \frac{1}{2}(c_1 + c_2),
\end{aligned}$$

and for Portfolio 2

$$
\begin{aligned}
w_1 \exp\left(H\left(Y_1\right)\right) + w_2 \exp\left(H\left(Y_2\right)\right) &= \frac{1}{3}\exp\left(H\left(\frac{3}{2}X_1\right) + \frac{2}{3}H\left(\frac{3}{4}X_2\right)\right) \\
&= \frac{1}{3} \times \frac{3}{2}\exp\left(H\left(X_1\right)\right) + \frac{2}{3} \times \frac{3}{4}\exp\left(H\left(X_2\right)\right) \\
&= \frac{1}{2}(c_1 + c_2).
\end{aligned}
$$

Therefore, we obtain

$$DD^*(P_1) = DD^*(P_2) = \frac{c_1 + c_2 - 2c_3}{c_1 + c_2}.$$