An inner product is a pair \( \langle f, g \rangle : X \times X \to F \) such that

1) \( \langle f, g \rangle = \overline{\langle g, f \rangle} \) (SYMMETRY)

2) \( \langle f, f \rangle \geq 0 \) (and \( \langle f, f \rangle = 0 \) iff \( f \equiv 0 \)) (POSITIVE DEFINITE)

3) \( \langle af + g, h \rangle = a \langle f, h \rangle + \langle g, h \rangle \) (LINEARITY)

Note: 1) \( f \) and \( g \) and \( h \) can be functions, vectors, matrices, or lots of other things depending on the circumstance.

2) The \( X \times X \to F \) part of the definition just means \( f \) and \( g \) both come from a set \( X \) (could be polynomials, vectors, etc.) and the inner product \( \langle f, g \rangle \) gives a number in \( F \).

The inner product may always give real numbers
   (called a real inner product)

Or could sometimes give complex numbers
   (called a complex inner product).

**Example:** The dot product of 2 vectors is an inner product.

We prove for \( \mathbb{R}^2 \). In this case the dot product is:

\[
\langle u, v \rangle = u_1v_1 + u_2v_2
\]

We need to check this satisfies our 3 rules above.

Check 1) \( \langle u, v \rangle = u_1v_1 + u_2v_2 = \langle v, u \rangle \). \( \checkmark \)

Check 2) \( \langle u, u \rangle = u_1u_1 + u_2u_2 = u_1^2 + u_2^2 \)

\( \geq 0 \) (and = 0 if and only if \( u_1 = 0 \) and \( u_2 = 0 \) \( \Rightarrow u = 0 \))

Check 3) \( \langle au + v, w \rangle = (au_1 + v_1)w_1 + (au_2 + v_2)w_2 = a\langle u, w \rangle + \langle v, w \rangle \). \( \checkmark \)

\( \ddot{\smile} \) as all 3 conditions are satisfied

\( \langle u, v \rangle = u_1v_1 + u_2v_2 \) is an inner product.
There are lots of other inner products. For example:
\[ \langle u, v \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 \] is an inner product on \( \mathbb{R}^3 \)
\[ \langle u, v \rangle = 4u_1 v_1 + u_2 v_2 + 4u_3 v_3 \] is an inner product on \( \mathbb{R}^3 \)
\[ \langle f, g \rangle = \int_0^1 f(x)g(x)dx \] is an inner product for real valued integrable functions.
\[ \langle f, g \rangle = f(0)g(0) + f(1)g(1) \] is an inner product for \( f, g \in P_1 \).

You can test these the same way as the example on the previous page.

Examples of not inner products.

1) Show \( \langle u, v \rangle = u_1 v_1 + u_2 v_2 \) is not an inner product.

Proof: This will not satisfy the linearity condition.

Let \( u = (\frac{1}{2}, \frac{1}{3}), \quad v = (\frac{1}{4}), \quad \omega = (1), \quad a = 2 \)

Then \( \langle au + v, \omega \rangle = \langle 2(\frac{1}{2}) + (\frac{1}{3}), (1) \rangle \)
\[ = \langle (\frac{5}{6}), (1) \rangle \]
\[ = 5 \times 1 + 8 \times 1^2 \]
\[ = 69 \]

But \( a \langle u, \omega \rangle + \langle u, \omega \rangle = 2 \langle (\frac{1}{2}), (1) \rangle + \langle (\frac{1}{3}), (1) \rangle \)
\[ = 2 \left( 1 \times 1 + 2 \times 1^2 \right) + \left( 3 \times 1 + 4 \times 1^3 \right) \]
\[ = 2 \times 5 + 19 \]
\[ = 29 \]

Our answers are not the same

\[ {\text{\small o: Linearity fails}} \]
\[ {\text{\small o: Not an inner product.}} \]

2) Show \( \langle u, v \rangle = u_1 v_1 + u_2 v_1 + u_2 v_2 \) is not an inner product.

Proof: This will not satisfy symmetry.

Let \( u = (\frac{1}{2}), \quad v = (\frac{3}{4}) \)

then \( \langle u, v \rangle = u_1 v_1 + u_2 v_2 = 1 \times 3 + 2 \times 3 + 2 \times 4 = 17 \)

and \( \langle v, u \rangle = v_1 u_1 + v_2 u_2 = 3 \times 1 + 4 \times 1 + 4 \times 2 = 15 \)

Not the same

\[ {\text{\small o: symmetry fails}} \]
\[ {\text{\small o: Not an inner product.}} \]
You should notice from the previous examples, there is a big difference in the method of proving something is an inner product, and proving something is not.

We also use the inner product definitions in the following example:

Example: Find \( \langle u + v, w - 2u \rangle \) given \( \langle u, v \rangle = 2i \), \( \langle u, w \rangle = 1 + i \), \( \langle v, w \rangle = 4 \) and \( ||u|| = 1 \)

(Note: \( ||u|| \) is defined by \( ||u|| = \sqrt{\langle u, u \rangle} \), also note this is the first complex inner product example)

Solution: by linearity property:

\[
\langle u + v, w - 2u \rangle = \langle u, w - 2u \rangle + \langle v, w - 2u \rangle
\]

\[
= \overline{\langle w - 2u, u \rangle} + \overline{\langle w - 2u, v \rangle} \quad \text{(symmetry)}
\]

\[
= \overline{\langle w, u \rangle} - 2\langle u, u \rangle + \overline{\langle w, v \rangle} - 2\langle v, v \rangle \quad \text{(linearity again)}
\]

\[
= \langle u, w \rangle - 2||u||^2 + \langle v, w \rangle - 2||v||^2 \quad \text{(symmetry again)}
\]

Now sub in:

\[
= 1 + i - 2 \times 1 + 4 - 2 \times 2i
\]

\[
= 1 + i - 2 + 4 + 4i
\]

\[
= 3 + 5i
\]

Projections

The projection of \( u \) onto \( v \) is given by:

\[
\text{proj}_v u = \frac{\langle u, v \rangle}{\langle v, v \rangle} v
\]

If \( v_1, v_2 \) are orthogonal then the projection of \( u \) onto \( V = \text{span} \{v_1, v_2\} \) is

\[
\text{proj}_V u = \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle u, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2
\]

The same pattern continues for more orthogonal vectors

\[
\text{proj}_V u = \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle u, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 + \ldots + \frac{\langle u, v_k \rangle}{\langle v_k, v_k \rangle} v_k
\]
Example: Given \( \langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx \), and the orthogonal set \( V = \{ 1, x, x^2 - \frac{1}{3} \} \) find \( \text{proj}_V e^x \).

**Solution:**

\[
\text{proj}_V e^x = \frac{\langle 1, e^x \rangle}{\langle 1, 1 \rangle} + \frac{\langle x, e^x \rangle}{\langle x, x \rangle} x + \frac{\langle x^2 - \frac{1}{3}, e^x \rangle}{\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \rangle} (x^2 - \frac{1}{3})
\]

We have 6 inner products to calculate:

1) \( \langle 1, 1 \rangle = \int_{-1}^{1} 1^2 \, dx = 2 \)

2) \( \langle x, x \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \)

3) \( \langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3} \rangle = \int_{-1}^{1} \left( x^2 - \frac{1}{3} \right)^2 \, dx = \int_{-1}^{1} x^4 - \frac{2}{3} x^2 + \frac{1}{9} \, dx = \frac{2}{5} - \frac{2}{3} + \frac{2}{9} \)

4) \( \langle 1, e^x \rangle = \int_{-1}^{1} e^x \, dx = e^x \bigg|_{-1}^{1} = e - e^{-1} = \frac{8}{5} \)

5) \( \langle x, e^x \rangle = \int_{-1}^{1} xe^x \, dx = xe^x - \int_{-1}^{1} e^x \, dx = xe^x - e^x \bigg|_{-1}^{1} = -2e^{-1} \)

6) \( \langle x^2 - \frac{1}{3}, e^x \rangle = \int_{-1}^{1} \left( x^2 - \frac{1}{3} \right) e^x \, dx = \int_{-1}^{1} x^2 e^x \, dx - \int_{-1}^{1} \frac{1}{3} e^x \, dx = x^2 e^x - 2xe^x \bigg|_{-1}^{1} - \frac{1}{3} e^x = x^2 e^x - 2xe^x + \int_{-1}^{1} 2e^x \, dx - \frac{1}{3} e^x = x^2 e^x - 2xe^x + 2e^{-1} \bigg|_{-1}^{1} \)

\[ = \frac{2}{5} e^{-1} - \frac{11}{3} e^{-1} \]

Sub into projection.

\[
\text{proj}_V e^x = \frac{\frac{8}{5} e - \frac{8}{5} e^{-1}}{2} + \frac{-2e^{-1}}{\frac{2}{5}} x + \frac{(\frac{2}{5} e^{-1} - \frac{11}{3} e^{-1})}{\frac{2}{5} \cdot \frac{2}{5}} (x^2 - \frac{1}{3})
\]

\[
= \frac{e - e^{-1}}{2} - 3e^{-1} x + \frac{15}{8} (2e^{-1} - 11e^{-1}) (x^2 - \frac{1}{3})
\]

You can interpret this as the closest polynomial in \( P_2 \) to \( e^x \).

The application is that if you project a difficult function onto an easy vector space (eg polynomials or sine curves etc) then you have an approximate function that is much easier to work with. The process above works for any inner product.
Gram–Schmidt Process

A Basis $\mathcal{B} = \{u_1, u_2, \ldots, u_n\}$ is orthogonal if $\langle u_i, u_j \rangle = 0$ for all $1 \leq i, j \leq n$ and $i \neq j$.

A Basis $\mathcal{B} = \{u_1, u_2, \ldots, u_n\}$ is orthonormal if $\langle u_i, u_j \rangle = \begin{cases} 1 & ; i=j \\ 0 & ; i \neq j \end{cases}$

The Gram Schmidt process creates an orthonormal basis from a regular basis.

Example: Find an orthonormal basis from $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ in $\mathbb{R}^3$.

Solution: First we check the given set is linearly independent:

$$a(\hat{e}_1) + b(\hat{e}_2) + c(\hat{e}_3) = (0,0,0)$$

put matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

row reduce $a(\hat{e}_1) + b(\hat{e}_2) + c(\hat{e}_3) = (0,0,0)$

$\Rightarrow a = b = c = 0$ is the only solution.

$\Rightarrow$ the set $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ is linearly independent.

$\Rightarrow$ We can use Gram Schmidt.

Method: From $\{u_1, u_2, u_3\}$ linearly independent

set $\tilde{\mathcal{V}}_1 = \mathcal{V}_1$, $\tilde{\mathcal{V}}_2 = u_2 - \text{proj}_{\tilde{\mathcal{V}}_1} u_2$, $\tilde{\mathcal{V}}_3 = u_3 - \text{proj}_{\tilde{\mathcal{V}}_2} u_3$.

then $\mathcal{V}_1 = \frac{\tilde{\mathcal{V}}_1}{\|\tilde{\mathcal{V}}_1\|}$, $\mathcal{V}_2 = \frac{\tilde{\mathcal{V}}_2}{\|\tilde{\mathcal{V}}_2\|}$, $\mathcal{V}_3 = \frac{\tilde{\mathcal{V}}_3}{\|\tilde{\mathcal{V}}_3\|}$ is an orthonormal basis.

(Note: Some people like to normalise as they go, this can make things easier, and can make them harder, it is up to you).

Example continued: So in our case:

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then $\tilde{\mathcal{V}}_1 = u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\tilde{\mathcal{V}}_2 = u_2 - \text{proj}_{\tilde{\mathcal{V}}_1} u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle}{\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \leftarrow \text{Note I can ignore the } \frac{1}{2} \text{ and take } \mathcal{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We were not given an inner product, and we have vectors, so we assume the dot product.
\[ \tilde{v}_3 = u_3 - \text{proj}_{u_2} u_3 - \text{proj}_{u_1} u_3 \]
\[ = (1) - \frac{\langle (1), (1) \rangle}{\langle (1), (1) \rangle} (1) - \frac{\langle (1), (1) \rangle}{\langle (1), (1) \rangle} \cdot \frac{\langle (1), (1) \rangle}{\langle (1), (1) \rangle} (1) \]
\[ = (1) - \frac{2}{2} (1) - \frac{2}{2} (-\frac{1}{2}) \]
\[ = \frac{1}{3} \left( -\frac{1}{2} \right) \]

**Similar to before I ignore the \( \frac{1}{3} \).**

Now we normalise.

\[ v_1 = \frac{\tilde{v}_1}{\| \tilde{v}_1 \|} = \frac{(1)}{\sqrt{\langle (1), (1) \rangle}} \]
\[ = \frac{1}{\sqrt{2}} (1) \]

\[ v_2 = \frac{\tilde{v}_2}{\| \tilde{v}_2 \|} = \frac{(-\frac{1}{2})}{\sqrt{\langle (-\frac{1}{2}), (-\frac{1}{2}) \rangle}} \]
\[ = \frac{1}{\sqrt{3}} (-\frac{1}{2}) \]

\[ v_3 = \frac{\tilde{v}_3}{\| \tilde{v}_3 \|} = \frac{(1)}{\sqrt{\langle (1), (1) \rangle}} \]
\[ = \frac{1}{\sqrt{3}} (1) \]

\[ \left\{ \frac{1}{\sqrt{2}} (1), \frac{1}{\sqrt{3}} (-\frac{1}{2}), \frac{1}{\sqrt{3}} (1) \right\} \]

**This step, we have to keep the coefficients, and can't ignore them.**

Form an orthonormal basis for \( \mathbb{R}^3 \).