Forecasting intraday financial time series with sieve bootstrapping and dynamic updating

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Intraday financial time series

1. CAPM considers returns using *low-frequency* spot prices, where price changes are ignored

[1] T. Andersen, T. Su, V. Todorov and Z. Zhang (2023+), Intraday periodic volatility curves, *Journal of the American Statistical Association*, in press

[2] D. Donoho and J. Tanner (2009), Observed universality of phase transitions in high-dimensional geometry, with implications for modern data analysis and signal processing, *Philosophical Transactions of the Royal Society A*, 367, 4273-4293

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Intraday financial time series

1. CAPM considers returns using *low-frequency* spot prices, where price changes are ignored
2. Intraday high-frequency[1] financial data take form of curves that can be sequentially observed over time
3. High-frequency data give rise to (dense) functional time series -> 'bless of dimensionality' [2]

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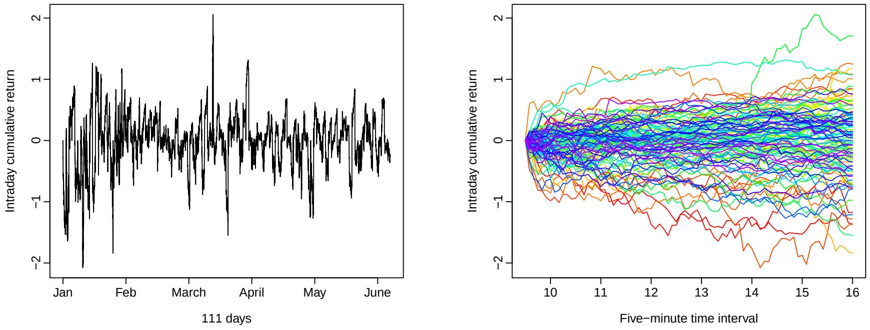
Examples of functional time series (FTS)



- A time series of functions is generated from a stochastic process $\mathcal{X}_t(u)$ where $u \in \mathcal{I} \subset R$, $t \in \mathcal{Z}$
- Modeling temporal dependence within & among functions

Advantages of functional time series

1. Study temporal correlation of an intraday functional object & learn about how correlation progress over days

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3. Study not only level but also derivatives[3] of functions –> dynamic modeling

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Road map

1. Introduce a functional time-series forecasting method for one-day-ahead prediction

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2. When partially observed data in most recent day becomes available, incorporate them to improve forecast accuracy
3. Apply a sieve bootstrap method for uncertainty quantification

Data

1. S&P/ASX All Ordinaries (XAO), 500 largest companies in Australian equities market

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4. 5-minute[4] intraday close prices of XAO from January 4 to December 23, 2021 from Refinitiv

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Cumulative intraday return (CIDR)

1. Let $P_t(u_i), t \in \mathbb{Z}_+, i = 2, \dots, \tau$, $\tau = 75$ be 5-minute close price of XAO at intraday time u_i between 10:00 & 16:10 Sydney time on day t

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4. For a stationary series, compute CIDR

$$\mathcal{X}_t(u_i) = 100 \times [\ln P_t(u_i) - \ln P_t(u_1)]$$

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5. Via inverse transformation,

$$P_t(u_i) = \exp^{\frac{\mathcal{X}_t(u_i)}{100}} \times P_t(u_1)$$

Functional principal component regression

1. For a time series of functions $[\mathcal{X}_1(u), \dots, \mathcal{X}_n(u)]$, mean function

$$\overline{\mathcal{X}}(u) = \frac{1}{n}\sum_{t=1}^{n} \mathcal{X}_t(u)$$

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2. Covariance function is

$$\text{cov}[\mathcal{X}(u), \mathcal{X}(v)] = \text{E}\{[\mathcal{X}(u) - \overline{\mathcal{X}}(u)][\mathcal{X}(v) - \overline{\mathcal{X}}(v)]\}$$

Mercer's lemma

Covariance function can be approximated by orthonormal eigenfunctions

$$\text{cov}[\mathcal{X}(u), \mathcal{X}(v)] = \sum_{k=1}^{\infty} \widehat{\lambda}_k \widehat{\phi}_k(u) \widehat{\phi}_k(v)$$

- $\widehat{\phi}_k(u)$: k^{th} orthonormal functional principal components
- $\widehat{\lambda}_k$: k^{th} eigenvalue

Karhunen-Loève expansion

1. Any functional realization $\mathcal{X}_t(u)$ can be expressed

$$\begin{aligned}\mathcal{X}_t(u) &= \overline{\mathcal{X}}(u) + \sum_{k=1}^{\infty} \underbrace{\widehat{\beta}_{t,k}}_{\langle \mathcal{X}_t(u) - \overline{\mathcal{X}}(u), \widehat{\phi}_k(u)\rangle} \widehat{\phi}_k(u) \\ &= \overline{\mathcal{X}}(u) + \sum_{k=1}^{K} \widehat{\beta}_{t,k}\widehat{\phi}_k(u) + e_t(u)\end{aligned}$$

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- K: retained number of principal components

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- K: retained number of principal components
- $e_t(u)$: error term

Eigenvalue ratio criterion

2. K is selected

$$K = \underset{1\leq k\leq k_{\max}}{\arg\min}\left\{\frac{\widehat{\lambda}_{k+1}}{\widehat{\lambda}_k}\times \mathbb{1}\left(\frac{\widehat{\lambda}_k}{\widehat{\lambda}_1}\geq \upsilon\right) + \mathbb{1}\left(\frac{\widehat{\lambda}_k}{\widehat{\lambda}_1} < \upsilon\right)\right\},$$

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- $\upsilon = 1/\ln[\max(\widehat{\lambda}_1, n)]$ is a pre-specified positive number
- $k_{\max} = \#\{k|\widehat{\lambda}_k \geq \sum_{k=1}^{n} \widehat{\lambda}_k/n\}$
- $\mathbb{1}\{\cdot\}$: binary indicator function.

VAR(p) model

3. Let $\widehat{\boldsymbol{\beta}} = (\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{\beta}}_2, \dots, \widehat{\boldsymbol{\beta}}_K)$

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4. VAR(p) model

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- $\widehat{\boldsymbol{A}}_{\xi,p}$: $(K \times K)$ coefficient matrix of forward score series
- $(\widehat{\boldsymbol{\epsilon}}_{p+1}, \dots, \widehat{\boldsymbol{\epsilon}}_n)$: residuals after fitting VAR(p) model to K-dimensional multivariate time series of scores

Order selection

1. Order p of VAR model can be chosen from AIC_c by minimizing

$$\text{AIC}_\text{c}(p) = n \ln |\widehat{\boldsymbol{\Sigma}}_{\widehat{\epsilon},p}| + \frac{n(nK + pK^2)}{n - K(p+1) - 1},$$

over a set of $p = \{1, 2, \ldots, 10\}$

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2. After fitting VAR(p), compute residuals $\widehat{\mathbf{\Sigma}}_{\widehat{\epsilon},p} = \frac{1}{n-p} \sum_{t=p+1}^{n} \widehat{\boldsymbol{\epsilon}}_t \widehat{\boldsymbol{\epsilon}}_t^\top$

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2. One-step-ahead forecast is

$$\begin{aligned}\widehat{\mathcal{X}}_{n+1|n}(u) &= \mathsf{E}[\mathcal{X}_{n+1}(u)|\boldsymbol{\mathcal{X}}(u), \overline{\mathcal{X}}(u), \boldsymbol{\Phi}(u)] \\ &= \overline{\mathcal{X}}(u) + \sum_{k=1}^{K} \widehat{\beta}_{n+1|n,k}\widehat{\phi}_k(u)\end{aligned}$$

where $\widehat{\beta}_{n+1|n,k}$: one-step-ahead prediction from VAR(p)

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   where $\widehat{\beta}_{n+1|n,k}$: one-step-ahead prediction from VAR(p)
3. If $K = 1$, VAR(p) reduces to AR(p)

Sieve bootstrap

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- $\{\varepsilon_t\}$: zero-mean i.i.d. innovation process with $\mathrm{E}\|\varepsilon_t\|^2 < \infty$

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   - $f : \mathcal{H}^\infty \to \mathcal{H}$
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2. Based on last ℓ observed functions, $\mathcal{X}_{n,\ell} = (\mathcal{X}_n, \mathcal{X}_{n-1}, \dots, \mathcal{X}_{n-\ell+1})$ for $\ell < n$, a predictor

$$\widehat{\mathcal{X}}_{n+1} = \widehat{g}(\mathcal{X}_n, \mathcal{X}_{n-1}, \dots, \mathcal{X}_{n-\ell+1})$$

where $\widehat{g} : \mathcal{H}^\ell \to \mathcal{H}$ estimated operator

Prediction error

3. Prediction error $\mathcal{E}_{n+1} = \mathcal{X}_{n+1} - \widehat{\mathcal{X}}_{n+1}$ given $\mathcal{X}_{n,\ell}$

$$\begin{aligned}\mathcal{E}_{n+1} &= \mathcal{X}_{n+1} - \widehat{\mathcal{X}}_{n+1} \\ &= \vartheta_{n+1} + [f(\mathcal{X}_n, \mathcal{X}_{n-1}, \dots) - g(\mathcal{X}_n, \mathcal{X}_{n-1}, \dots, \mathcal{X}_{n+1-\ell}] + \\ &\quad [g(\mathcal{X}_n, \mathcal{X}_{n-1}, \dots, \mathcal{X}_{n+1-\ell}) - \widehat{g}(\widehat{\mathcal{X}}_n, \widehat{\mathcal{X}}_{n-1}, \dots, \widehat{\mathcal{X}}_{n+1-\ell})] \\ &= \mathcal{E}_{I,n+1} + \mathcal{E}_{M,n+1} + \mathcal{E}_{E,n+1}\end{aligned}$$

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- $\mathcal{E}_{I,n+1}$: error attributable to i.i.d. innovation

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4 Ultimate goal: Prediction band $[\widehat{\mathcal{X}}_{n+1}(u) - L_n(u), \widehat{\mathcal{X}}_{n+1}(u) + U_n(u)]$

$$\lim_{n\rightarrow\infty} \text{Pr}(\widehat{\mathcal{X}}_{n+1}(u) - L_n(u) \leq \mathcal{X}_{n+1}(u) \leq \widehat{\mathcal{X}}_{n+1}(u) + U_n(u), \forall u \in \mathcal{I} | \mathcal{X}_{n,\ell}) = 1 - \alpha$$

VAR(p) forward series

1. Sieve bootstrap uses VAR(p) to generate forward score forecasts

$$\boldsymbol{\beta}^*_{n+1} = \sum_{\xi=1}^{p} \widehat{\boldsymbol{A}}_{\xi,p}\boldsymbol{\beta}^*_{n+1-\xi} + \boldsymbol{\epsilon}^*_{n+1}$$

where $\boldsymbol{\beta}^*_{n+1-\xi} = \widehat{\boldsymbol{\beta}}_{n+1-\xi}$ for $n+1-\xi \leq n$

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2. $\boldsymbol{\epsilon}^*_{n+1}$: i.i.d. resampled from centered residuals $\{\widehat{\epsilon}_t - \overline{\epsilon}, t = p+1,\dots,n\}$

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3. Compute

$$\mathcal{X}^*_{n+1}(u) = \overline{\mathcal{X}}(u) + \sum_{k=1}^{K} \beta^*_{n+1,k}\widehat{\phi}_k(u) + e^*_{n+1}(u)$$

VAR(p) forward series

1. Sieve bootstrap uses VAR(p) to generate forward score forecasts

$$\boldsymbol{\beta}^*_{n+1} = \sum_{\xi=1}^{p} \widehat{\boldsymbol{A}}_{\xi,p}\boldsymbol{\beta}^*_{n+1-\xi} + \boldsymbol{\epsilon}^*_{n+1}$$

   where $\boldsymbol{\beta}^*_{n+1-\xi} = \widehat{\boldsymbol{\beta}}_{n+1-\xi}$ for $n+1-\xi \leq n$
2. $\boldsymbol{\epsilon}^*_{n+1}$: i.i.d. resampled from centered residuals $\{\widehat{\epsilon}_t - \overline{\epsilon}, t = p+1, \dots, n\}$
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1. Because of stationarity, VAR(p) can go backward in time to generate bootstrap samples of scores

$$\widehat{\boldsymbol{\beta}}_t = \sum_{\xi=1}^{p} \widehat{\boldsymbol{B}}_{\xi,p} \widehat{\boldsymbol{\beta}}_{t+\xi} + \boldsymbol{\eta}_t$$

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- $\boldsymbol{I}_K$: $(K \times K)$ diagonal matrix

VAR(p) bootstrap scores

1. Bootstrap samples for backward series

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$$\boldsymbol{\beta}_t^* = \sum_{\xi=1}^{p} \widehat{\boldsymbol{B}}_{\xi,p}\boldsymbol{\beta}_{t+\xi}^* + \boldsymbol{\eta}_t^*$$

2. Bootstrap functional time series

$$\mathcal{X}_t^*(u) = \overline{\mathcal{X}}(u) + \sum_{k=1}^{K} \beta_{t,k}^* \widehat{\phi}_k(u) + e_t^*(u)$$

where $e_t^*(u)$: i.i.d. resampled from $\{e_t(u) - \overline{e}(u)\}$

Anyone has a favor method

FAR(1)

$$\widehat{\mathcal{X}}_{n+1} = \overline{\mathcal{X}}(u) + \gamma[\mathcal{X}_n(u) - \overline{\mathcal{X}}(u)]$$

where γ: bounded linear operator, measuring first-order autocorrelation

$$\widehat{\gamma} = \frac{\widehat{\Gamma}(1)}{\widehat{\Gamma}(0)}$$

$$\widehat{\Gamma}(0) = \frac{1}{n}\sum_{t=1}^{n}[\mathcal{X}_t(u) - \overline{\mathcal{X}}(u)] \otimes [\mathcal{X}_t(u) - \overline{\mathcal{X}}(u)]$$

$$\widehat{\Gamma}(1) = \frac{1}{n}\sum_{t=1}^{n-1}[\mathcal{X}_t(u) - \overline{\mathcal{X}}(u)] \otimes [\mathcal{X}_{t+1}(u) - \overline{\mathcal{X}}(u)]$$

Model calibration error

1. Distribution of prediction error $\mathcal{E}^*_{n+1}(u) = \mathcal{X}^*_{n+1}(u) - \widehat{\mathcal{X}}^*_{n+1}(u)$: proxy for distribution of $\mathcal{E}_{n+1}(u) = \mathcal{X}_{n+1}(u) - \widehat{\mathcal{X}}_{n+1}(u)$ given $[\mathcal{X}_{n-\ell+1}(u), \dots, \mathcal{X}_n(u)]$

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Prediction band

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2. $(1-\alpha)$ uniform prediction band for $\mathcal{X}_{n+1}(u)$ is

$$\left[\widehat{\mathcal{X}}_{n+1}(u) - Q^*_{1-\alpha}\sigma^*_{n+1}(u), \widehat{\mathcal{X}}_{n+1}(u) + Q^*_{1-\alpha}\sigma^*_{n+1}(u)\right]$$

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1. When a functional time series is formed as segments of a univariate time series, most recent curve is observed sequentially

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3. Update forecasts in remainder of day $n+1$, $\mathcal{X}_{n+1}(u_l)$, $u_l \in (u_m, u_\tau]$

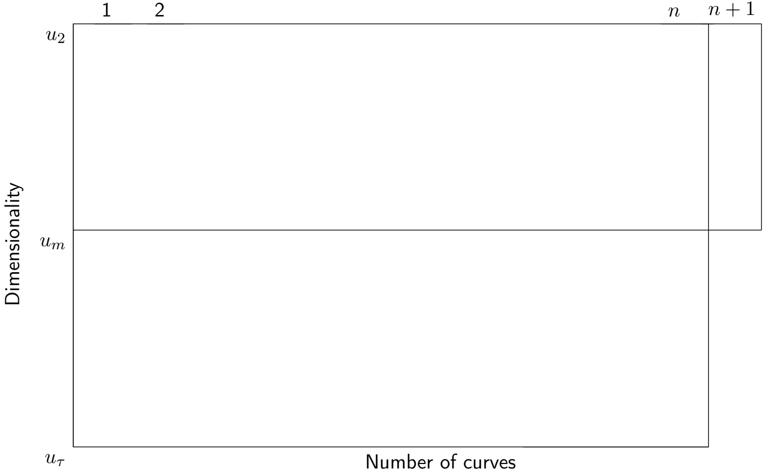

Figure: *Conceptual diagram of dynamic updating.*

Penalized least squares (PLS) method

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3. PLS regression coefficient estimates minimize a penalized residual sum of squares

$$\underset{\boldsymbol{\beta}_{n+1}}{\arg\min}\Big\{[\mathcal{X}^c_{n+1}(u_e) - \boldsymbol{\mathcal{F}}_e\boldsymbol{\beta}_{n+1}]^\top[\mathcal{X}^c_{n+1}(u_e) - \boldsymbol{\mathcal{F}}_e\boldsymbol{\beta}_{n+1}] + \lambda(\boldsymbol{\beta}_{n+1} - \widehat{\boldsymbol{\beta}}^{\text{TS}}_{n+1|n})^\top(\boldsymbol{\beta}_{n+1} - \widehat{\boldsymbol{\beta}}^{\text{TS}}_{n+1|n})\Big\}$$

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- $\lambda \in (0, \infty)$: shrinkage parameter
- $\boldsymbol{\mathcal{F}}_e$: $(m \times K)$ matrix, whose $(i,k)^{\text{th}}$ entry is $\widehat{\phi}_k(u_i)$ for $2 \leq i \leq m$

PLS regression coefficient

1. By taking first derivative with respect to $\boldsymbol{\beta}_{n+1}$

$$\widehat{\boldsymbol{\beta}}_{n+1}^{\text{PLS}} = (\boldsymbol{\mathcal{F}}_e^\top \boldsymbol{\mathcal{F}}_e + \lambda \boldsymbol{I}_K)^{-1} \left[\boldsymbol{\mathcal{F}}_e^\top \mathcal{X}_{n+1}^c(u_e) + \lambda \widehat{\boldsymbol{\beta}}_{n+1|n}^{\text{TS}}\right]$$

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2. When shrinkage parameter

$$\widehat{\boldsymbol{\beta}}^{\text{PLS}}_{n+1} = \begin{cases} \widehat{\boldsymbol{\beta}}^{\text{OLS}}_{n+1} & \text{if } \lambda \to 0; \\ \widehat{\boldsymbol{\beta}}^{\text{TS}}_{n+1|n} & \text{if } \lambda \to \infty; \\ (\widehat{\boldsymbol{\beta}}^{\text{OLS}}_{n+1}, \widehat{\boldsymbol{\beta}}^{\text{TS}}_{n+1|n}) & \text{if } 0 < \lambda < \infty. \end{cases}$$

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3. With optimal λ, PLS forecasts of $\mathcal{X}_{n+1}(u_l)$

$$\widehat{\mathcal{X}}^{\text{PLS}}_{n+1}(u_l) = \overline{\mathcal{X}}(u_l) + \sum_{k=1}^{K} \widehat{\beta}^{\text{PLS}}_{n+1,k} \widehat{\phi}_k(u_l)$$

Selection of λ

Split data into a training set, a validation set, a testing set

$1:150$	$151:200$	$201:250$
Training	Validation	Testing

Updating interval forecasts

1. Bootstrap B samples of TS forecast regression coefficient,

$$\widehat{\boldsymbol{\beta}}^{*,\text{TS}}_{n+1|n} = (\widehat{\beta}^{*,\text{TS}}_{n+1|n,1}, \dots, \widehat{\beta}^{*,\text{TS}}_{n+1|n,K})^\top$$

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2. For each $b = 1, \dots, B = 400$

$$\widehat{\mathcal{X}}^{*,\text{PLS}}_{n+1} = \overline{\mathcal{X}}(u_l) + \sum_{k=1}^{K} \widehat{\beta}^{*,\text{PLS}}_{n+1,k} \widehat{\phi}_k(u_l) + e^*_{n+1}(u_l)$$

where $e^*_{n+1}(u_l)$: bootstrapped residuals for updating period

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where $e^*_{n+1}(u_l)$: bootstrapped residuals for updating period

3. $(1-\alpha)$ PIs for updated forecasts are: $\alpha/2$ & $(1-\alpha/2)$ quantiles of

$$\left\{\widehat{\mathcal{X}}^{1,\text{PLS}}_{n+1}(u_l), \dots, \widehat{\mathcal{X}}^{B,\text{PLS}}_{n+1}(u_l)\right\}$$

Function-on-function linear regression

1. Regression

$$\mathcal{X}_{n+1}^{l}(u) = \overline{\mathcal{X}}^{l}(u) + \int [\mathcal{X}_{n+1}^{e}(v) - \overline{\mathcal{X}}^{e}(v)]\beta(u,v)dv + \xi_{n+1}^{l}(u)$$

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- $v \in [u_2, u_m]$ & $u \in (u_m, u_\tau]$: function support ranges for observed & updating periods

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- $\mathcal{X}_{n+1}^{e}(v)$ & $\mathcal{X}_{n+1}^{l}(u)$: functional predictor & response
- $\beta(u,v)$: bivariate regression coefficient function

FPCA

$$\mathcal{X}_t^e(v) = \overline{\mathcal{X}}^e(v) + \sum_{r=1}^{\infty} \widehat{\theta}_{t,r}\widehat{\phi}_r^e(v) = \overline{\mathcal{X}}^e(v) + \sum_{r=1}^{R} \widehat{\theta}_{t,r}\widehat{\phi}_r^e(v) + \kappa_t^e(v)$$

$$\mathcal{X}_t^l(u) = \overline{\mathcal{X}}^l(u) + \sum_{s=1}^{\infty} \widehat{\vartheta}_{t,s}\widehat{\phi}_s^l(u) = \overline{\mathcal{X}}^l(u) + \sum_{s=1}^{S} \widehat{\vartheta}_{t,s}\widehat{\phi}_s^l(u) + \delta_t^l(u)$$

Ordinary least squares (OLS)

1. To estimate $\beta(u,v)$, let $\widehat{\boldsymbol{\theta}} = [\widehat{\boldsymbol{\theta}}_1, \widehat{\boldsymbol{\theta}}_2, \dots, \widehat{\boldsymbol{\theta}}_R]$, $\widehat{\boldsymbol{\vartheta}} = [\widehat{\boldsymbol{\vartheta}}_1, \widehat{\boldsymbol{\vartheta}}_2, \dots, \widehat{\boldsymbol{\vartheta}}_S]$

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2. Via OLS, linear relationship between $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\vartheta}}$

$$\widehat{\boldsymbol{\vartheta}} = \widehat{\boldsymbol{\theta}} \times \boldsymbol{\rho}$$

$$\widehat{\boldsymbol{\rho}} = (\widehat{\boldsymbol{\theta}}^\top \widehat{\boldsymbol{\theta}})^{-1} \widehat{\boldsymbol{\theta}}^\top \widehat{\boldsymbol{\vartheta}}$$

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3. One-step-ahead forecast of $\mathcal{X}_{n+1}^l(u)$

$$\widehat{\mathcal{X}}_{n+1}^l(u) = \overline{\mathcal{X}}^l(u) + \sum_{s=1}^{S} \widehat{\vartheta}_{n+1,s} \widehat{\phi}_s^l(u)$$
$$\approx \overline{\mathcal{X}}^l(u) + \widehat{\boldsymbol{\theta}}_{n+1} \times \widehat{\boldsymbol{\rho}} \times \widehat{\boldsymbol{\phi}}^l(u)$$

Updating interval forecasts

1. One-step-ahead interval forecast of $\mathcal{X}^l_{n+1}(u)$ is

$$\widehat{\mathcal{X}}^{l,*}_{n+1}(u) = \overline{\mathcal{X}}^l(u) + \int [\mathcal{X}^e_{n+1}(v) - \overline{\mathcal{X}}^e(v)]\widehat{\beta}^*(u,v)dv + e^{l,*}_{n+1}(u)$$

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   - $\widehat{\beta}^*(u,v)$: bootstrap regression coefficient estimates

Updating interval forecasts

1. One-step-ahead interval forecast of $\mathcal{X}^l_{n+1}(u)$ is

$$\widehat{\mathcal{X}}^{l,*}_{n+1}(u) = \overline{\mathcal{X}}^l(u) + \int [\mathcal{X}^e_{n+1}(v) - \overline{\mathcal{X}}^e(v)]\widehat{\beta}^*(u,v)dv + e^{l,*}_{n+1}(u)$$

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   - $e_{n+1}^{l,*}(u)$: bootstrap residuals for updating period

2. Via sieve bootstrap, obtain bootstrap curves, and then estimate $\widehat{\beta}^{*}(u,v)$
3. $(1-\alpha)$ PI for updated forecasts are $\alpha/2$ & $(1-\alpha/2)$ quantiles of $\{\widehat{\mathcal{X}}_{n+1}^{l,1}(u), \dots, \widehat{\mathcal{X}}_{n+1}^{l,B}(u)\}$

Expanding-window scheme

1. Initial training samples are curves from Days 1 to 200, compute one-day-ahead forecast

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3. Iterate this procedure until training samples cover entire 250 days

Mean squared forecast error

1. MSFE measures closeness of forecasts compared with actual values of variable being forecast

$$\text{MSFE}(u_i) = \frac{1}{n_{\text{test}}}\sum_{\iota=1}^{n_{\text{test}}}\left[\mathcal{X}_\iota(u_i) - \widehat{\mathcal{X}}_\iota(u_i)\right]^2$$

$$\text{MSFE} = \frac{1}{\tau - 1}\sum_{i=2}^{\tau}\text{MSFE}(u_i)$$

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- $\mathcal{X}_{\iota}$: holdout samples for i^{th} intraday period on ι day
- $n_{\text{test}} = 50$: number of curves in forecasting period

Empirical coverage probability (ECP)

$$\text{ECP}_{\text{pointwise}} = 1 - \frac{1}{n_{\text{test}} \times (\tau - 1)} \sum_{\iota=1}^{n_{\text{test}}} \sum_{i=2}^{\tau} \left[\mathbb{1}\{\mathcal{X}_\iota(u_i) < \widehat{\mathcal{X}}_\iota^{\text{lb}}(u_i)\} + \mathbb{1}\{\mathcal{X}_\iota(u_i) > \widehat{\mathcal{X}}_\iota^{\text{ub}}(u_i)\} \right]$$

$$\text{ECP}_{\text{uniform}} = 1 - \frac{1}{n_{\text{test}}} \sum_{\iota=1}^{n_{\text{test}}} \left[\mathbb{1}\{\mathcal{X}_\iota(u) < \widehat{\mathcal{X}}_\iota^{\text{lb}}(u)\} + \mathbb{1}\{\mathcal{X}_\iota(u) > \widehat{\mathcal{X}}_\iota^{\text{ub}}(u)\} \right].$$

Uniform prediction bands are wider than pointwise PIs

Interval score[5]

$$
\begin{aligned}
S_\alpha\left[\widehat{\mathcal{X}}_\iota^{\text{lb}}(u_i), \widehat{\mathcal{X}}_\iota^{\text{ub}}(u_i), \mathcal{X}_\iota(u_i)\right] = &\left[\widehat{\mathcal{X}}_\iota^{\text{ub}}(u_i) - \widehat{\mathcal{X}}_\iota^{\text{lb}}(u_i)\right] \\
&+ \frac{2}{\alpha}\left[\widehat{\mathcal{X}}_\iota^{\text{lb}}(u_i) - \mathcal{X}_\iota(u_i)\right] \mathbb{1}\left\{\mathcal{X}_\iota(u_i) < \widehat{\mathcal{X}}_\iota^{\text{lb}}(u_i)\right\} \\
&+ \frac{2}{\alpha}\left[\mathcal{X}_\iota(u_i) - \widehat{\mathcal{X}}_\iota^{\text{ub}}(u_i)\right] \mathbb{1}\left\{\mathcal{X}_\iota(u_i) > \widehat{\mathcal{X}}_\iota^{\text{ub}}(u_i)\right\}.
\end{aligned}
$$

[5] T. Gneiting and A. E. Raftery (2007) Strictly proper scoring rules, prediction, and estimation, *Journal of the American Statistical Association*, 102(477), 359-378

Mean interval score

Averaged over different days in forecasting period, mean interval score

$$\overline{S}_\alpha(u_i) = \frac{1}{n_{\text{test}}}\sum_{\iota=1}^{n_{\text{test}}} S_\alpha\left[\widehat{\mathcal{X}}_\iota^{\text{lb}}(u_i), \widehat{\mathcal{X}}_\iota^{\text{ub}}(u_i), \mathcal{X}_\iota(u_i)\right]$$

$$\overline{S}_\alpha = \frac{1}{\tau-1}\sum_{i=2}^{\tau}\overline{S}_\alpha(u_i).$$

An illustration

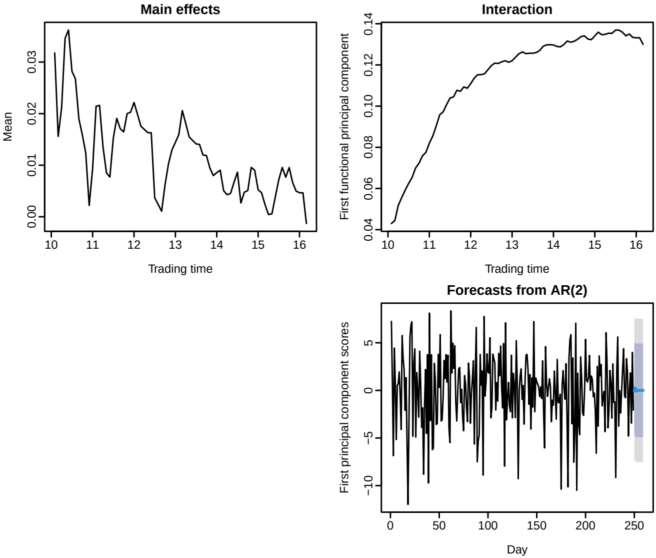
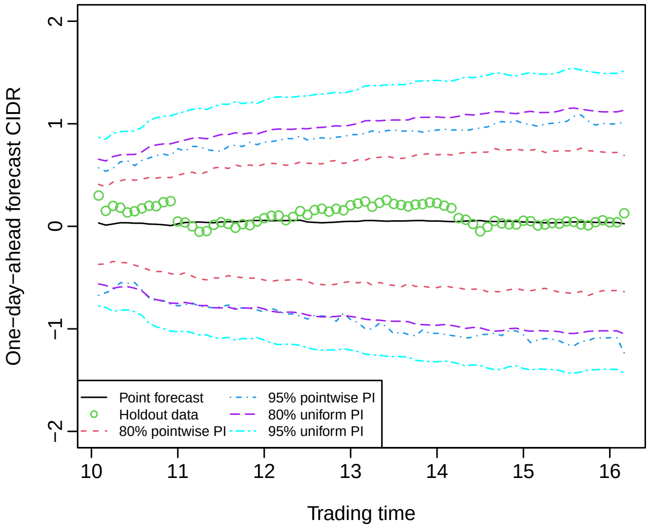


Figure: *From January 4 to December 22, 2021, forecast CIDR for December 23*

Forecast for December 23



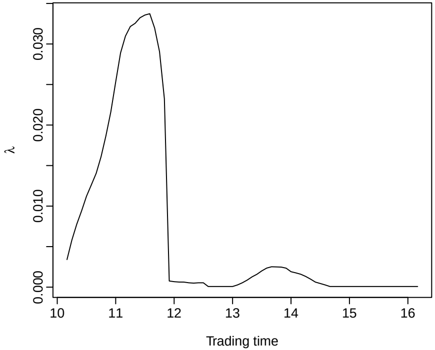
Point forecast accuracy

Averaging over 50 days in forecasting period & 73 different intraday updating periods

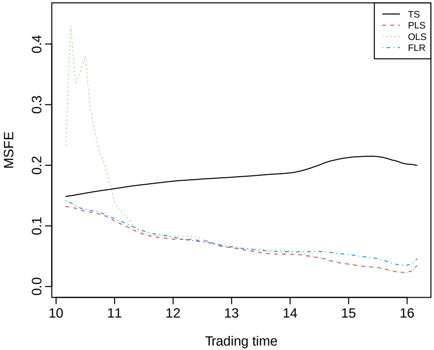
		$\text{ECP}_{\text{pointwise},1-\alpha}$		$\text{ECP}_{\text{uniform},1-\alpha}$		$\overline{S}_\alpha$	
Method	MSFE	$\alpha = 0.2$	$\alpha = 0.05$	$\alpha = 0.2$	$\alpha = 0.05$	$\alpha = 0.2$	$\alpha = 0.05$
TS	0.1474	0.89	0.98	0.88	0.96	1.43	2.08

Updating point forecasts

PLS method has best point forecast accuracy



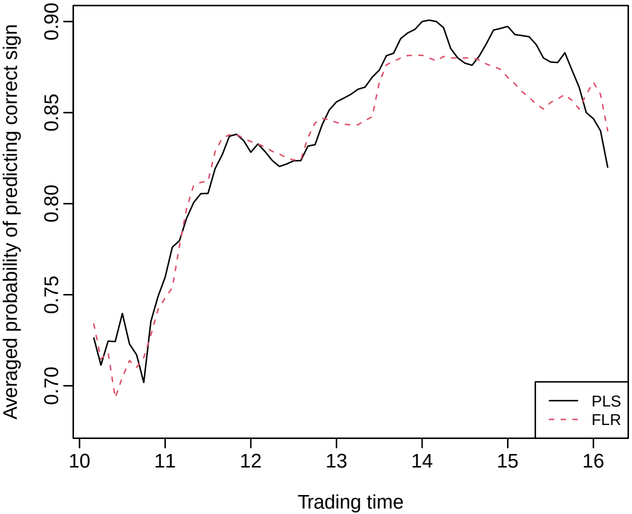
(a) *Selected optimal λ values*



(b) *Out-of-sample MSFE*

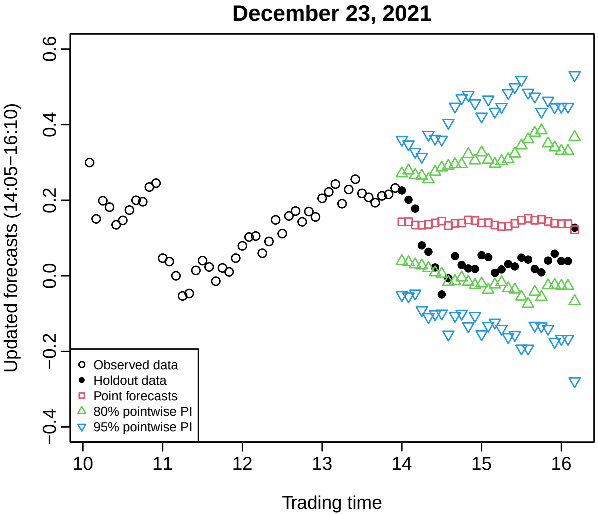
Predicting signs

CIDRs are hard to predict, but signs of future values are easier

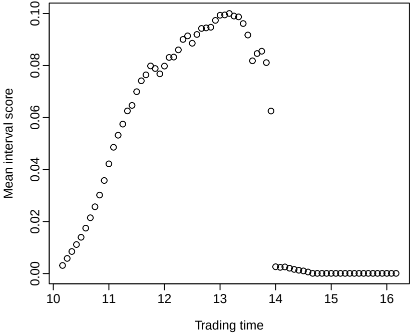


Updating prediction intervals

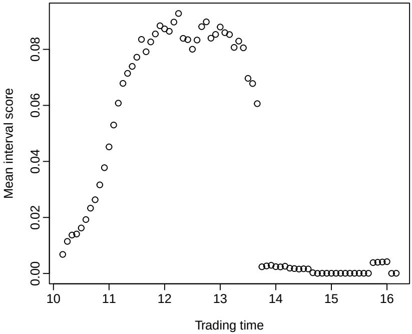
As we observe new data from beginning to 14:00, apply PLS with optimal λ to update point & interval forecast



Estimated optimal λ

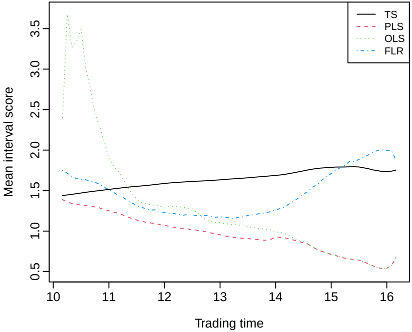


(c) *80% nominal coverage*

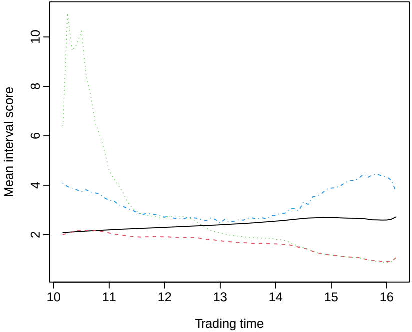


(d) *95% nominal coverage*

Mean interval score



(e) *80% nominal coverage*



(f) *95% nominal coverage*

Updating forecast accuracy

		$\text{ECP}_{\text{pointwise},1-\alpha}$		$\overline{S}_\alpha$	
Method	MSFE	$\alpha = 0.2$	$\alpha = 0.05$	$\alpha = 0.2$	$\alpha = 0.05$
TS	0.1838	0.8817	0.9688	1.6429	2.4277
PLS	0.0672	0.7275	0.8851	0.9564	1.6440
FLR	0.0735	0.5001	0.7266	1.4661	3.2729

Future research

1. Consider other sampling frequencies

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2. Outliers can affect estimation of covariance, one can use robust FPCA
3. With validation samples, PLS parameters can be adaptively chosen without recomputing

Thank you

Paper: https://onlinelibrary.wiley.com/doi/full/10.1002/for.3000

RG: https://www.researchgate.net/profile/Han-Lin-Shang