Workshop Topic:

1. Limits and continuity

We start by using $\varepsilon$-$\delta$ to prove a limit exists.

**Ex:** Prove $\lim_{(x,y) \to (4,0)} \frac{x^5 y^2}{(x^2 + y^2)^3} = 0$ by $\varepsilon$-$\delta$ definition.

**Solution:** We show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\sqrt{x^2 + y^2} < \delta$ implies $|f(x) - L| < \varepsilon$.

That is, $|f(x) - L| = \left| \frac{x^5 y^2}{(x^2 + y^2)^3} - 0 \right|

= \left| \frac{x^5 y^2}{(x^2 + y^2)^3} \right|

\leq \left| \frac{(x^2 + y^2)\frac{x^5 y^2}{(x^2 + y^2)^3}}{(x^2 + y^2)} \right| \quad \text{use this step that} \quad |x| \leq \sqrt{x^2 + y^2} \quad \text{and} \quad |y| \leq \sqrt{x^2 + y^2}

\leq \frac{(x^2 + y^2)^{3/2}}{2}

\leq \delta^2

\leq \varepsilon \quad \text{choosing} \delta = \varepsilon^{1/3}

Note, always choose $\delta$, never choose $\varepsilon$ when proving limit exists.
Next a 1 dimensional example:

Ex: Prove by $\varepsilon$-$\delta$ that the function $f(x) = x^4$ satisfies
\[ \lim_{x \to 2} f(x^2) = 4. \]

Solution: For any $\varepsilon > 0$ there exists $\delta = \min \left( \frac{\varepsilon}{16}, 1 \right) > 0$ such that $|x - 2| < \delta$ implies:
\[
|f(x) - 4| = |x^4 - 4| \\
= |x - 2| \cdot |x^2 + 2| \\
< \delta \cdot |x^2 + 2| \\
< \delta (4 + \delta) \\
< \frac{\varepsilon}{16} \quad \text{if} \quad \delta < 1.
\]

$\therefore$ The limit $= 4$

It is often easier to prove (in 2D) that limits exist by polar coordinates.

Ex: Prove
\[ \lim_{(x,y) \to (0,0)} \frac{x^3 y^2}{(x^2 + y^2)^2} = 0 \]

by polar coordinates.

Solution: Let $x = r \cos \theta$, $y = r \sin \theta$.

Then
\[ \lim_{(x,y) \to (0,0)} = \lim_{r \to 0} \frac{r^3 \cos^3 \theta \cdot r^2 \sin^2 \theta}{(r^2)^2} \]
\[ = \lim_{r \to 0} \frac{r \cos^3 \theta \cdot \sin^2 \theta}{r^4} \]
\[ = \lim_{r \to 0} \frac{r \cos^3 \theta \cdot \sin^2 \theta}{r^4} \]
\[ = 0 \quad \text{by squeeze principle.} \]

Note:
1) $x^2 + y^2 = r^2$ simplifies denominator
2) NEVER take $\theta \to 0$
3) It is important that $\cos^3 \theta \cdot \sin^2 \theta$ at the end is bounded above by 1 and you must point this out.
Now we look at when a limit does not exist.

Ex: Prove \( \lim_{(x,y) \to (x_0,y_0)} \frac{x^2y^2}{(x^2+y^2)^3} \) does not exist

Solution: Let \( y = x \)

Then \( \lim_{x \to 0} f(x,y) = \lim_{x \to 0} \frac{x^4}{(2x^2)^3} \)

\[ = \lim_{x \to 0} \frac{x^4}{8x^6} \]

\[ = \lim_{x \to 0} \frac{1}{8} \]

\[ = \frac{1}{8} \]

Limit: Not the same, \( \therefore \) limit does not exist.

Next let \( x = 0 \)

Then \( \lim_{y \to 0} f(0,y) = \lim_{y \to 0} \frac{(0)^2y^2}{(0^2+y^2)^3} \)

\[ = \lim_{y \to 0} \frac{0}{y^6} \]

\[ = \lim_{y \to 0} 0 \]

\[ = 0 \]

Note: Do NOT use this method of checking directions to prove a limit exists.

You can also use polar coordinates to prove a limit does not exist. For the question above we would say:

Let \( x = r \cos \theta, \ y = r \sin \theta \) (Then \( x^2 + y^2 = r^2 \))

And: \( \lim_{(x,y) \to (x_0,y_0)} f(x,y) = \lim_{r \to 0^+} \frac{(r \cos \theta)^2(r \sin \theta)^2}{(r^2)^3} \) (if they exist)

\[ = \lim_{r \to 0^+} \frac{r^4 \cos^2 \theta \sin^2 \theta}{r^6} \]

\[ = \lim_{r \to 0^+} \cos^2 \theta \sin^2 \theta \]

This limit does not exist as it is different for different values of \( \theta \),

For example \( \theta = 0 \Rightarrow \lim = 0 \)

but \( \theta = \frac{\pi}{4} \Rightarrow \lim = \frac{1}{4} \).

\( \therefore \) Limit does not exist.
Continuity

A function $f$ is continuous at a point $a$ if

$$\lim_{x \to a} f(x) = f(a)$$

Ex: Prove $f(x) = \begin{cases} \frac{x^3y^2}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$

Solution: From an earlier example we saw that

$$\lim_{(x,y) \to (0,0)} \frac{x^3y^2}{(x^2+y^2)^2} = 0$$

$\therefore$ As $\lim_{(x,y) \to (0,0)} f(x,y) = 0 = f(0,0)$

the function $f$ is continuous at $(0,0)$.

Note: If we had not found $\lim_{(x,y) \to (0,0)} f(x,y)$ earlier, we would have to do that here.

So to answer a question on continuity, we just find the limit as usual, then check if the limit equals $f(a)$.
The next example looks at finding the limit of a composition \( \lim_{(x,y) \to (0,0)} f(g(x,y)) \).

Ex: Find the limit of the function \( f(x,y) = \frac{\cos(\frac{\pi xy}{2})}{xy-1} \) as \( (x,y) \to (0,0) \).

Solution: Let \( g(x,y) = xy \), then \( g \) is continuous, and \( \lim_{(x,y) \to (0,0)} g(x,y) = 0 \).

We can now say: \( \lim_{(x,y) \to (0,0)} f(x,y) = \lim_{g \to 0} \frac{\cos(\frac{\pi g}{2})}{g-1} \)

\[ = \lim_{g \to 1} -\frac{\pi}{2} \sin(\frac{\pi g}{2}) \text{ by L'Hopital's rule} \]

\[ = -\frac{\pi}{2} \sin(\frac{\pi}{2}) \]

\[ = -\frac{\pi}{2}. \]

Another example on continuity:

Ex: Find \( a \) and \( b \) such that the function \( f(x,y) = \begin{cases} xe^{-y} + ay & ; x \geq y \\ b+3x & ; x < y \end{cases} \)

is continuous.

Solution: The two halves of the function; \( xe^{-y} + ay \) and \( b+3x \) are continuous. Therefore, we only need to check where the two halves meet (on the line \( x=y \)).

On the line \( x=y \) we have:

\[ f(y,y) = ye^{-y} + ay = (1+a)y \]

From below the line \( x=y \) we have

\[ f(x,y) \to b+3y \]

The two parts need to be equal for continuity

\[ (1+a)y = b+3y, \]

compare constants: \( b = 0 \)

compare \( y ' s: a = \frac{b}{x} \)

\( \Rightarrow a = -3 \) and \( b = 0 \) ensure continuity.
We finish with a difficult example on continuity

Ex: Find \( c \in \mathbb{R} \) such that \( f(xy) = \begin{cases} \frac{ye^{x-1} - y}{(x-1)^2 + y^2} & (x,y) \neq (1,0) \quad \text{Note not} \quad (0,0) \\ c & (x,y) = (1,0) \end{cases} \)

Solution: First we check if \( \lim_{(x,y) \to (1,0)} f(xy) \) exists.

\[
\lim_{(x,y) \to (1,0)} f(xy) = \lim_{(x,y) \to (1,0)} \left| \frac{ye^{x-1} - y}{(x-1)^2 + y^2} \right| \quad \text{use polar coordinates}
\]

\[
= \lim_{r \to 0} \left| \frac{r \sin \theta e^{r \cos \theta} - r \sin \theta}{r} \right| \quad \text{Note: extra } +1 \text{ in } x \text{ part as } (1,0) \text{ is the centre here.}
\]

\[
= \lim_{r \to 0} \left| \frac{r \sin \theta (e^{r \cos \theta} - 1)}{r} \right| \quad \text{Use the squeeze principle:}
\]

\[
-1 \leq \cos \theta \leq 1 \\
-r \leq r \cos \theta \leq r \quad \text{both sides by } r \\
e^{-r} \leq e^{r \cos \theta} \leq e^{r} \\
e^{-r-1} \leq e^{r \cos \theta} - 1 \leq e^{-r-1} \\
|e^{r \cos \theta} - 1| \leq e^{-r-1} \\
|\sin \theta (e^{r \cos \theta} - 1)| \leq e^{-r-1} \quad \text{absolute value} \\
\lim_{r \to 0} |\sin \theta (e^{r \cos \theta} - 1)| \leq \lim_{r \to 0} e^{-r-1} \quad \text{take limits}
\]

\[
\leq 1 - 1 \\
\leq 0.
\]

\[\therefore \lim_{r \to 0} |\sin \theta (e^{r \cos \theta} - 1)| = 0\]

\[\therefore \lim_{(x,y) \to (1,0)} f(xy) = 0.\]

\[\therefore \text{choose } c = 0 \text{ to get } (x,y) \to (1,0) \quad f(xy) = f(1,0) \text{ to make } f(xy) \text{ continuous.}\]

End of Workshop

Please post any questions on the iLearn forums.