Default Risk Propagation in Multilayer Systems under Common Shock

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February 2023

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Motivation:

- Investigate default risk propagation within a multilayer system
- Develop a multilayer system featuring the dependence structure between the layers of the financial system (interdependence) as well as within each layer (intradependence).
  - Consider for simplicity a two-layer network of intermediaries in the low-default environment
  - Discover strong propagation of default risk across different layers
- The proposed framework has a wide range of applications across financial, insurance, reinsurance, climate and other sectors of the economy.

Motivation:

- Examples of multilayer networks
  - Financial system: interconnectedness of its institutions, which are linked through networks of different types of financial contracts.
  - Insurance and the rest of the economy: Insurance companies invest in capital markets; banks are also constituents of the capital market. In addition to their firm-specific risks, they are exposed to market volatility and correlations.
  - Insurance and reinsurance: traditional risk management strategy for insurers, reinsurance deepens the interconnections among insurance companies and thus aggravates the systemic risk.
  - Climate networks: A multilayer network could be constructed from temperature measurements, precipitation, pressure, wind, humidity, and cloudiness on different sites in the world. Climate extremes, compound events can be regarded as defaults which lead to default risk propagation within a multilayer system.

Model in brief:

- Consider a simplified system consisting of two layers/sub-systems $\boldsymbol{X}$ and $\boldsymbol{Y}$, of sizes d_X and d_Y, respectively – each composed of intermediaries of high-quality and hence low default risk. Such a model allows intradependence within each layer as well as interdependence between the two layers.
- probability of default (PD) $0 < p < 1$,
- Study conditional probability, interpreted as a measure of the propagation, of the layer $\boldsymbol{X}$ to incur a significant number of defaults, $N_X(p)$, given that the layer $\boldsymbol{Y}$ incurs a significant number of defaults, $N_Y(p)$
- Result: conditional probability is asymptotic to cp as $p \to 0$ with c representing the propagation effect across the two layers.
- Main result – a sharp asymptotic formula that, as $p \downarrow 0$,

$$P\left(N_X(p) \geq n_X | N_Y(p) \geq n_Y\right) = (c + o(1))\, p; \tag{1}$$

Introduction:

- Consider general intermediary Z that summarizes its rating migration and possible default.
- Probability of default (PD) is $0 < p < 1$, the intermediary defaults if and only if the latent variable Z is larger than its Value at Risk (VaR) at level $q = 1 - p$, defined by

$$VaR_q(Z) = \inf\left\{z \in \mathbb{R} : P\left(Z \leq z\right) \geq q\right\}, \qquad 0 < q < 1.$$

This standard threshold model approach to default originates from Merton's firm-value model; see Merton 1974 for the origin of this approach and Altman Et al. (2004) for a detailed review of the history of credit risk models. For the setting in terms of large credit portfolios, refer to Bassamboo et al (2008), Duffie and Lando (2001) and Tang et al. (2019).

Problem formulation:

- Consider a complex system consisting of multiple intermediaries – restrict our study to a system of two layers/sub-systems $\boldsymbol{X}$ and $\boldsymbol{Y}$, each consisting of intermediaries $D_X = \{1, \ldots, d_X\}$ and $D_Y = \{1, \ldots, d_Y\}$
- $\boldsymbol{X}$ and $\boldsymbol{Y}$ exhibit intradependence within their respective layers.
- $\boldsymbol{X}$ and $\boldsymbol{Y}$ also exhibit interdependence between the two layers.
- Study the probability that a significant number of defaults incurred in layer $\boldsymbol{Y}$ triggers a significant number of defaults in the layer $\boldsymbol{X}$.
- The number of defaults corresponding to default probability p

$$N_X(p) = \sum_{i=1}^{d_X} 1_{(X_i > VaR_q(X_i))}$$

$$N_Y(p) = \sum_{j=1}^{d_Y} 1_{(Y_j > VaR_q(Y_j))}.$$

Problem formulation:

- First objective: The intermediaries X_i's within each $\mathbf{X}$ are clearly dependent
- How to capture the dependency between intermediaries within this layers/sub-system?
- Use d_X-variate Archimedean copulas with generator $h_{\theta_1}(x)$ to model dependency between the components of $\mathbf{X}$ (Notation: $N_{\mathbf{X}}(p)|\theta_1$ where θ_1 is the dependence parameter)
- Second objective: capture the dependency between $N_{\mathbf{X}}(p)$ and $N_{\mathbf{Y}}(p)$, since defaults are common to several layers
- Assume that $\mathbf{Y}$ has the same generator as $\mathbf{X}$, but is indexed by θ_2, $h_{\theta_2}(x)$
- How to effectively capture interdependence?

Solution provided in this paper:

- Use Archimedean copulas with generators $h_{\theta_1}(x)$ and $h_{\theta_2}(y)$ to capture the dependency within $\mathbf{X}$, and $\mathbf{Y}$, respectively.
- Assume that a bivariate vector or copula parameters (θ_1, θ_2) follows jointly a bivariate generalised Pareto distribution.
- Describe conditional probability of default $P(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y)$ (when p is small)
- Important theoretical result:

$$\lim_{p \to 0} \frac{1}{p} P\left(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y\right) = c \tag{2}$$

- Coefficient c measures the propagation effect across the two layers

Outline:

- Introduction and motivation ✓
- Main results for the distribution of the number of losses $N_{\mathbf{X}}(p)$, moments calculation using copulas for one credit portfolio
- Introduce Archimedean and survival copulas to determine the distribution for the number of defaults (moments, asymptotic behaviour)
- Introduce multivariate generalised Pareto distribution that is required for modelling the dependency between two credit portfolios
- Bivariate case using Archimedean Copulas and Generalised Pareto Distribution: derivation of $P(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y)$ and its limiting behaviour
- Conclusion

Main results for the number of defaults $N_{\mathbf{X}}(p)$:

- Tail probability:

$$P(X_i > VaR_q(X_i)) = P(F_i(X_i) \geq 1-p) = P\left(\overline{F_i}(X_i) \leq p\right),$$

- The number of defaults $N(p)$

$$N(p) = \sum_{i=1}^{d} 1_{(X_i > VaR_q(X_i))} = \sum_{i=1}^{d} 1_{(\overline{F_i}(X_i) \leq p)}, \qquad 0 < p < 1.$$

- Assume that $(X_1, ..., X_d)$ posses a copula C with survival copula $\widehat{C}$. Probability for the number of defaults

$$P(N(p) = n) \;=\; \binom{d}{n} \sum_{k=0}^{d-n} (-1)^k \binom{d-n}{k} \widehat{C}(v_1 = \cdots = v_{n+k} = p)$$

where $\widehat{C}(\cdot)$ is a survival copula.

Recall: $F(x_1, ...x_d) = C(F_1(x_1), .., F_d(x_d))$ where $C(\cdot)$ is copula; then $\overline{F}(x_1, ...x_d) = \widehat{C}(\overline{F}(x_1), .., \overline{F}(x_d)) = \widehat{C}(v_1, ..., v_d)$.

Main results for the number of losses $N_{\mathbf{X}}(p)$: Moments calculation

$$E(N(p)) \quad = \quad E(\sum_{i=1}^{d} 1_{\left(\overline{F}_i(X_i) \leq p\right)}) = \sum_{i=1}^{d} P\left(\overline{F}_i(X_i) \leq p\right) = \sum_{i=1}^{d} \widehat{C}(v_i = p) = dp$$

$$\begin{aligned} E(N(p)^2) &= E\left[\left(\sum_{i=1}^{d} 1_{\left(\overline{F}_i(X_i) \leq p\right)}\right)^2\right] \\ &= dp + d(d-1)\widehat{C}(p,p), \end{aligned} \tag{3}$$

where we used the symmetry of $\widehat{C}$

$$VaR(N(p)) \quad = \quad dp + d(d-1)\widehat{C}(p,p) - (dp)^2$$

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Archimedean and survival copulas:

- $h(x)$ is a generator of an Archimedean survival copula that is *multiply monotonic* function of order d, i.e. condition

$$(-1)^k h^{(k)}(x) \geq 0 \tag{4}$$

holds for $k = 0, ..., d$, and $x \in \mathbf{R}_+$

- $h(x)$ satisfies

$$h(0) = 1, \tag{5}$$

$$\lim_{x \to \infty} h(x) = 0. \tag{6}$$

- Assuming that $h^{-1}(x)$ is well defined, the Archimedean survival copula can be defined as

$$\widehat{C}(v_1, ..., v_d) = h(h^{-1}(v_1) + h^{-1}(v_2) + \cdots + h^{-1}(v_d)). \tag{7}$$

Archimedean and survival copulas:

- This copula is completely symmetric and it holds

$$\begin{aligned}\widehat{C}(v) &= v, \\ \widehat{C}(v,v) &= h(2h^{-1}(v)), \\ &\vdots \\ \widehat{C}(\underbrace{v,\dots,v}_{d \text{ elements}}) &= h(dh^{-1}(v)).\end{aligned}$$

- Probability for the number of defaults

$$P\left(N(p)=n\right)=\binom{d}{n}\sum_{k=0}^{d-n}(-1)^k\binom{d-n}{k}h((n+k)h^{-1}(p)).$$

- Expectation

$$E(N(p))=dp$$

Archimedean and survival copulas:

- Variance

$$\begin{aligned} Var(N(p)) &= dp + d(d-1)\widehat{C}(p,p) - (dp)^2 \\ &= dp(1-p) - dp^2(d-1) + d(d-1)h(2h^{-1}(p)), \end{aligned}$$

where we used $\widehat{C}(p,p) = h(h^{-1}(p) + h^{-1}(p)) = h(2h^{-1}(p))$.

- **Proposition:**
Suppose that $h(x,\theta) \in C(\mathbb{R}_+ \times \overline{\Theta})$, and $h(x,\theta)$ is strictly decreasing function on x for all $\theta \in \overline{\Theta}$, where $\overline{\Theta}$ be a closure of Θ, and $h^{-1}(x,\theta)$ is continuous function on θ. Consider the following condition:

  ($\star$) *There exist a point $\theta_0 \in \overline{\Theta}$ (point of independence) such that $\lim_{\theta\to\theta_0} h(x) = a^{-x}$ for $a > 1$ and any given $x > 0$.*

  Then the condition $(\star)$ is necessary and sufficient for

$$\lim_{\theta\to\theta_0} Var(N_\theta(p)) \to dp(1-p) \tag{8}$$

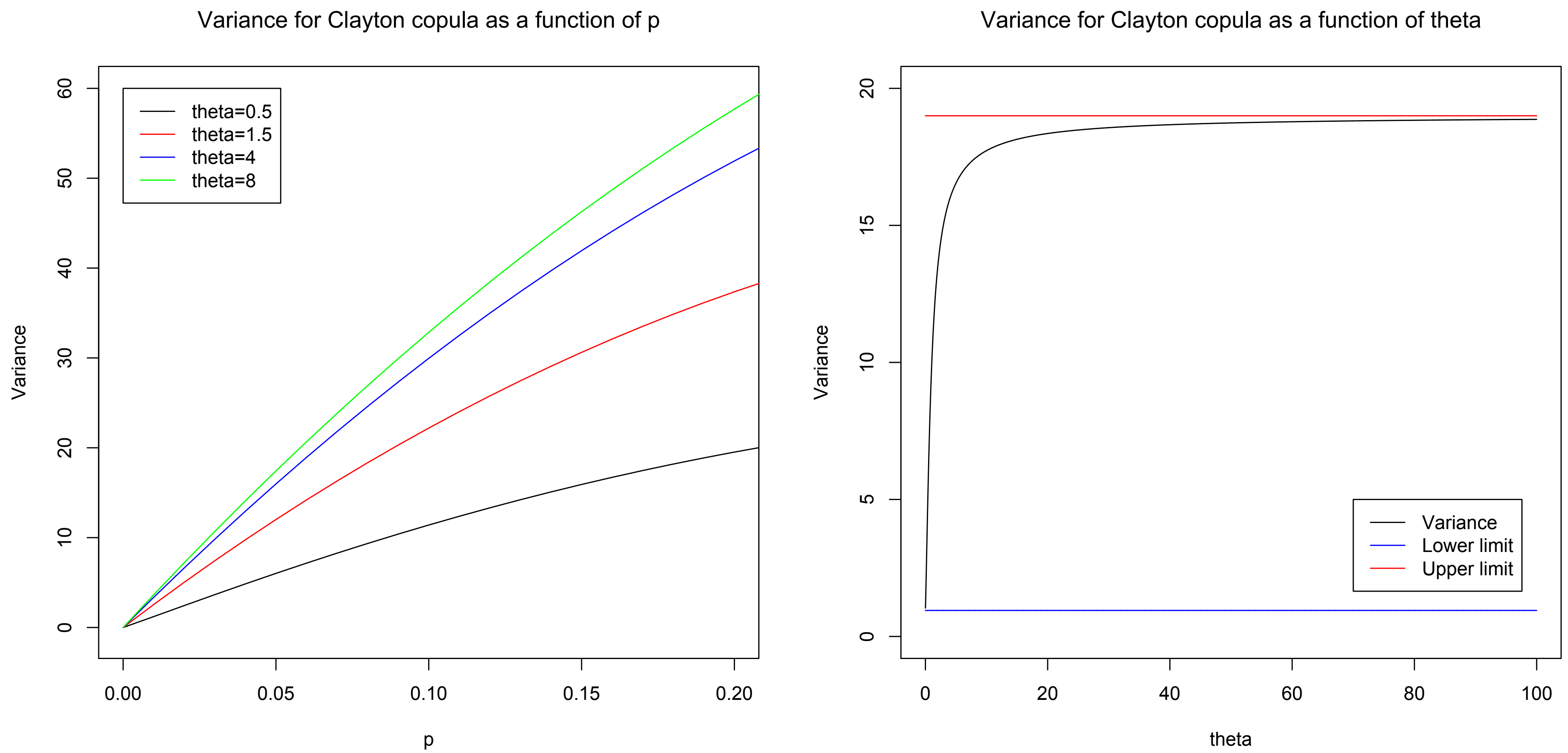
  for any given $p \in (0,1)$.

Archimedean and survival copulas

Copula	$h(x)$	θ_0, θ_∞	$Var(N(p))$	$\lim_{\theta\to\theta_\infty} Var(N(p))$
Clayton	$(1+\theta x)^{-\frac{1}{\theta}}$ $x \in [0,\infty), \theta \in [0,\infty)$	0, ∞	$dp(1-p) - d(d-1)p^2$ $+d(d-1)\left(1+2(p^{-\theta}-1)\right)^{-\frac{1}{\theta}}$	$dp(1-p) - d(d-1)p^2$ $+d(d-1)p$
Gumbel	$\exp(-x^{\frac{1}{\theta}})$ $x \in [0,\infty), \theta \in [1,\infty)$	1, ∞	$dp(1-p) - d(d-1)p^2$ $+d(d-1)p^{2^{\frac{1}{\theta}}}$	$dp(1-p) - dp^2(d-1)$ $+d(d-1)p$
Joe	$1-(1-\exp(-x))^{\frac{1}{\theta}}$ $x \in [0,\infty), \theta \in [1,\infty)$	1, ∞	$dp(1-p) - dp^2(d-1)$ $+d(d-1)\left(1-\left(2(1-p)^\theta-(1-p)^{2\theta}\right)^{\frac{1}{\theta}}\right)$	$dp(1-p) - dp^2(d-1)$ $+d(d-1)p$
Frank	$1-(1-\exp(-x))^{\frac{1}{\theta}}$ $x \in [0,\infty), \theta \in [0,\infty)$	0 ∞	$dp(1-p) - dp^2(d-1)$ $+d(d-1)\left(-\frac{1}{\theta}\ln\left(1+\frac{(\exp(-\theta p)-1)^2}{\exp(-\theta)-1}\right)\right)$	$dp(1-p) - dp^2(d-1)$ $+d(d-1)(1-p)$
AMH	$\frac{1-\theta}{\exp(x)-\theta}$ $x \in [0,\infty), \theta \in [0,1)$	0 1	$dp(1-p) - dp^2(d-1)$ $+d(d-1)\frac{1-\theta}{\left(\frac{1-\theta}{p}+\theta\right)^2-\theta}$	$dp(1-p) - dp^2(d-1)$ $-d(d-1)\frac{p}{p-2}$

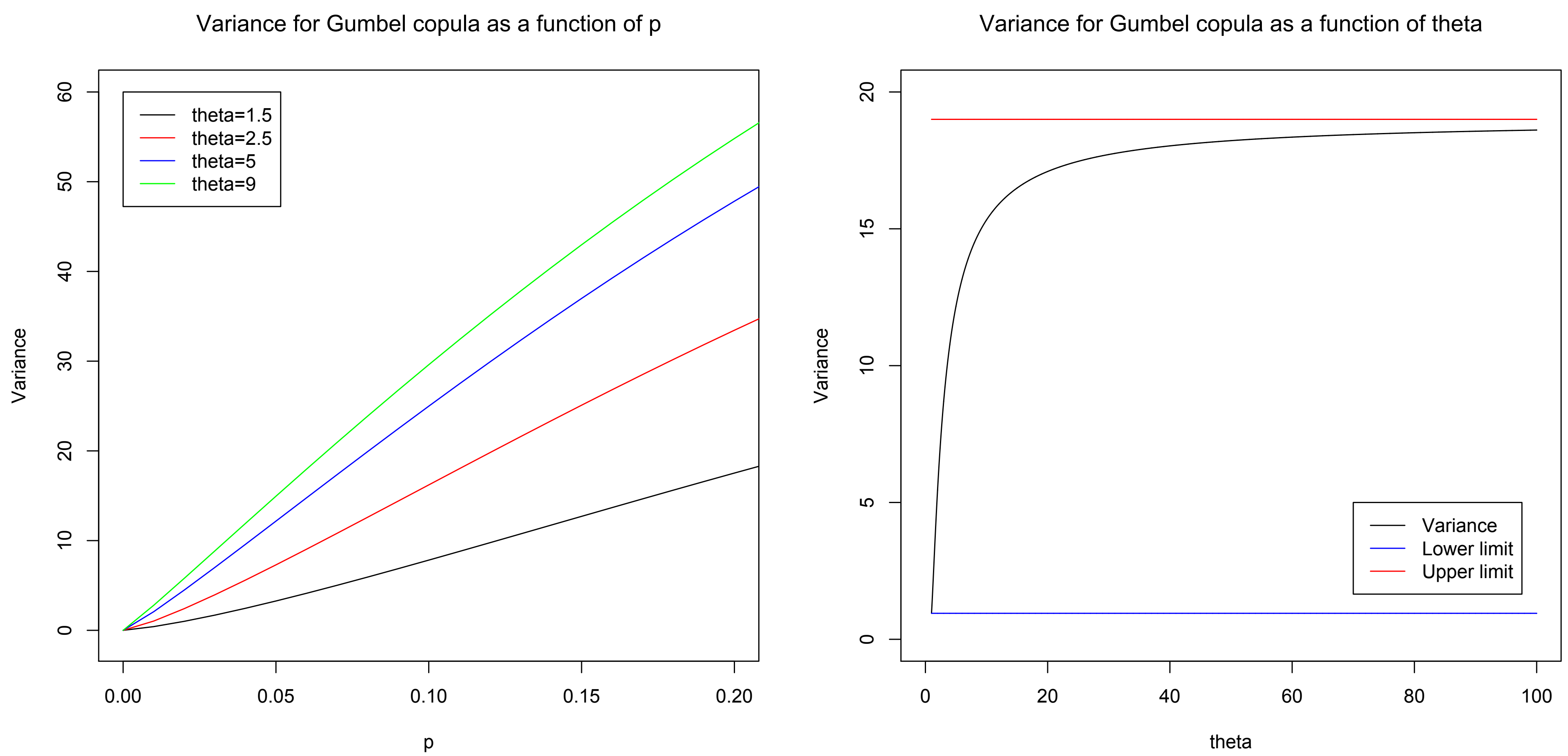
Note. Copula specifications: the generator function $h(x)$, the point of independence θ_0 and the point of maximal dependence θ_∞, variance for the number of defaults $Var(N(p))$ and variance for the number of defaults when the maximal dependence is achieved $\lim_{\theta\to\theta_\infty} Var(N(p))$. The variance for the number of defaults at the point of independence $\lim_{\theta\to\theta_0} Var(N(p)) = dp(1-p)$ for all copulas.

Variance for Archimedean copulas: Clayton



Left panels: Variance of Clayton copula as a function of p, and varying θ ($d = 20$ is fixed). Right panel: Variance of Clayton copula as a function of θ, its lower limit $\lim_{\theta \to \theta_0} Var(N(p))$ and upper limit $\lim_{\theta \to \theta_\infty} Var(N(p))$ ($d = 20, p = 0.05$ are fixed).

Variance for Archimedean copulas: Gumbel



Left panels: Variance of Gumbel copulas as a function of p, and varying θ ($d = 20$ is fixed). Right panel: Variance of Gumbel copulas as a function of θ, its lower limit $\lim_{\theta \to \theta_0} Var(N(p))$ and upper limit $\lim_{\theta \to \theta_\infty} Var(N(p))$ ($d = 20, p = 0.05$ are fixed).

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Multivariate generalised Pareto distribution

- Hendriks and Landsman (2017) introduce a class of multivariate generalised Pareto (GP) distributions (special cases: multivariate Pareto and multivariate Weibull distributions). There is a relationship between Archimedean survival copulas and GP distribution:

- Given a multivariate vector $\mathbf{X} = (X_1, ..., X_d)$ with marginal distributions for $X_i \geq 0$ for $i = 1, ..., d$ characterised via the survival functions

$$P(X_i > x_i) = h(\lambda_i x_i) = p_i, \tag{9}$$

$\lambda_i > 0$, any copula generator $h(\cdot)$ corresponds to a GP distribution as follows:

$$\begin{aligned} \overline{F}_{\mathbf{X}}(x_1, ..., x_d) &= P(X_1 > x_1, ..., X_n > x_n) \\ &= \widehat{C}\left(P(X_1 > x_1), ..., P(X_d > x_d)\right) \\ &= \widehat{C}(p_1, ..., p_d) \\ &= h(h^{-1}(p_1) + ... + h^{-1}(p_d)) \\ &= h(h^{-1}(h(\lambda_1 x_1)) + ... + h^{-1}(h(\lambda_d x_d))) \\ &= h(\lambda_1 x_1 + ... + \lambda_d x_d). \\ &= h\left(\sum_{i=1}^{d} \lambda_i x_i\right), \qquad x_i \geq 0. \end{aligned} \tag{10}$$

Multivariate generalised Pareto distribution

- If a cdf of $\mathbf{X}$ is characterised by $F_{\mathbf{X}} = 1 - \overline{F}_{\mathbf{X}}$, the joint pdf of $\mathbf{X}$ is

$$f_{\mathbf{X}}(x_1, ..., x_d) = (-1)^d \lambda_1 \cdots \lambda_d h^{(d)} \left(\sum_{i=1}^{d} \lambda_i x_i \right) \text{ with } x_i \geq 0. \tag{11}$$

- Marginal survival functions of X_i:

$$P(X_i > x_i) = P(X_1 > 0, ..., X_{i-1} > 0, X_i > x_i, X_{i+1} > 0, ...X_d > 0). \tag{12}$$

- Generator of multivariate Pareto (special case of GP):

$$H(x) = \frac{1}{(1+x)^{\alpha}} \text{ for } x \geq 0, \alpha > 0. \tag{13}$$

- Associated variance:

$$Var(N(p)) = dp(1-p) + d(d-1)p \left(\frac{1}{(2 - p^{1/\alpha})^{\alpha}} - p \right). \tag{14}$$

Multivariate generalised Pareto distribution

- For the multivariate Pareto distribution α captures heaviness of tail: When $\alpha \to \infty$ the tail becomes lighter
- The asymptote

$$\lim_{\alpha \to \infty} Var(N(p)) = dp(1-p). \tag{15}$$

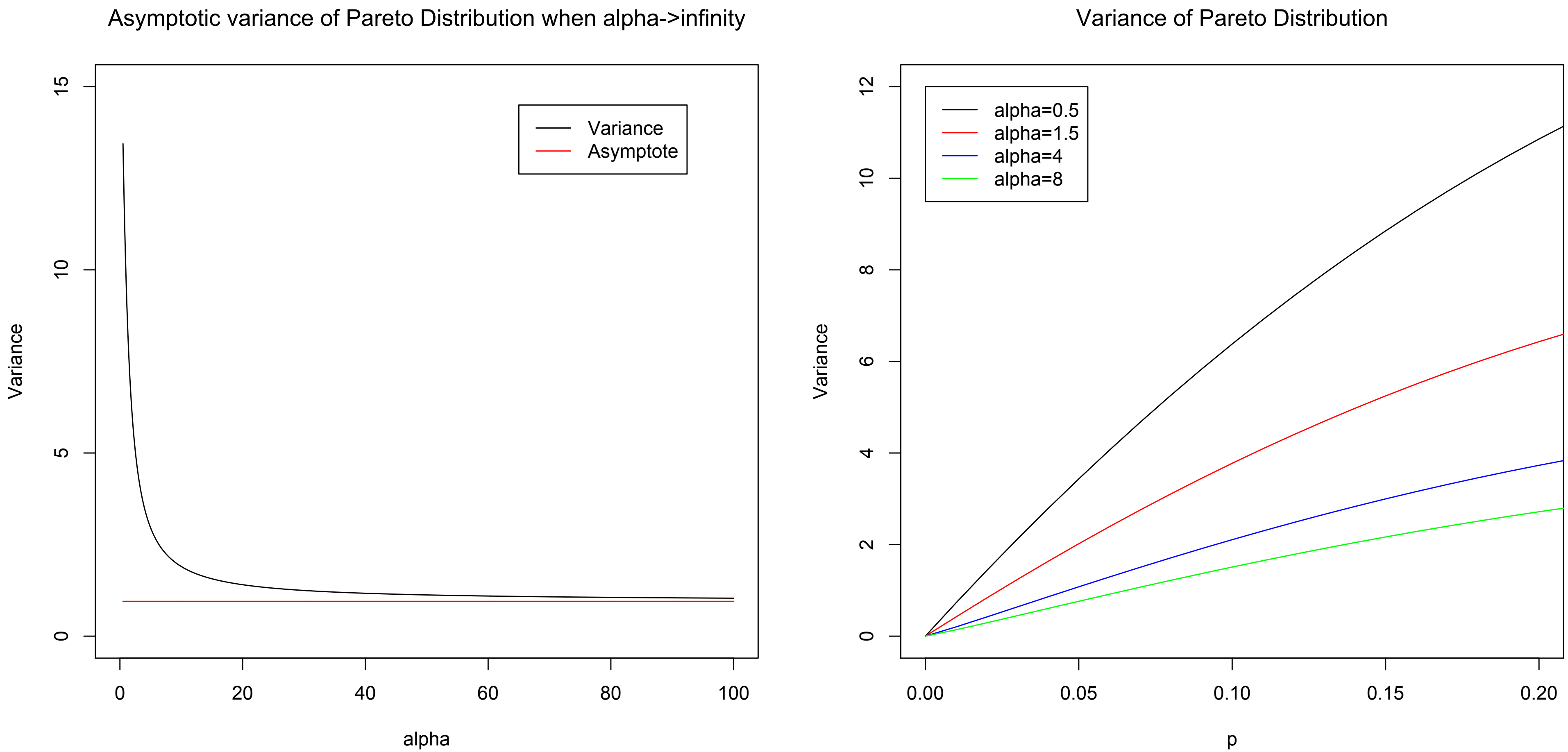
- Correlation between two bivariate Pareto random variables $(X_i, X_j), i \neq j$

$$Corr(X_i, X_j) = \frac{1}{\alpha},$$

- $Corr(X_i, X_j) \downarrow 0$ when $\alpha \uparrow \infty$ (independence between X_i and X_j)
- When $\alpha \downarrow 0$, we obtain maximal dependence, and it holds:

$$\lim_{\alpha \to 0} Var(N(p)) = \infty.$$

Variance of Pareto distribution



Left panel: Variance of Pareto distribution for fixed $d = 20$, fixed $p = 0.05$ and varying α. Right Panel: Variance of Pareto distribution for fixed $d = 10$, $\alpha \in \{0.5, 1.5, 4, 8\}$ and varying p.

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Bivariate Case using Archimedean Copulas and Generalised Pareto Distribution

- $\mathbf{X} = (X_1, ..., X_{d_X})$ is d_X-dimensional; $\mathbf{X}|\theta_1 \sim$ Archimedean copula with generator h_{θ_1}
- $\mathbf{Y} = (Y_1, ..., Y_{d_Y})$ is d_Y-dimensional; $\mathbf{Y}|\theta_2 \sim$ Archimedean copula with generator h_{θ_2}
- Assume that $\mathbf{X}|\theta_1$ and $\mathbf{Y}|\theta_2$ are conditionally independent
- Aim: using properties of the generator $h_\theta(x)$ (regularly varying or rapidly varying), derive
  - $P(N_{\mathbf{X}}(p) = n_1 | N_{\mathbf{Y}}(p) = n_2)$
  - $P(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y)$

Conditional probability of default

$$
\begin{aligned}
& P\left(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y\right) \\
= & \underbrace{\sum_{n_1=n_X}^{d_X} \sum_{n_2=n_Y}^{d_Y} P\left(\left(N_{\mathbf{X}}(p)=n_1\right) \cap \left(N_{\mathbf{Y}}(p)=n_2\right)\right)}_{A} \div \underbrace{\sum_{n_2=n_Y}^{d_Y} P(N_Y(p)=n_2)}_{B}
\end{aligned}
$$

- Assume $N_{\mathbf{X}}(p)|\theta_1$ and $N_{\mathbf{Y}}(p)|\theta_2$ are independent;

$$
\begin{aligned}
& P((N_{\mathbf{X}}(p)=n_1) \cap (N_{\mathbf{Y}}(p)=n_2)) \\
= & \int\int_{\Theta_1 \times \Theta_2} P\left(N_{\mathbf{X}}(p)=n_1|\theta_1\right) \times P\left(N_{\mathbf{Y}}(p)=n_2|\theta_2\right) \times \pi(\theta_1,\theta_2) d\theta_1 d\theta_2
\end{aligned}
$$

where the bivariate GP density $\pi(\theta_1, \theta_2)$

$$\pi(\theta_1,\theta_2) = \lambda_1\lambda_2 H^{(2)}\left(\lambda_1\theta_1 + \lambda_2\theta_2\right). \tag{15}$$

due to

$$(\theta_1,\theta_2) \sim H(\lambda_1\theta_1 + \lambda_2\theta_2), \tag{15}$$

Conditional probability of default

$$
\begin{aligned}
&P((N_{\mathbf{X}}(p) = n_1) \cap (N_{\mathbf{Y}}(p) = n_2)) \\
= \; & \int_{\Theta_1} \int_{\Theta_2} \binom{d_X}{n_1} \sum_{k=0}^{d_X - n_1} (-1)^k \binom{d_X - n_1}{k} h_{\theta_1}((n_1 + k) h_{\theta_1}^{-1}(p)) \\
& \times \binom{d_Y}{n_2} \sum_{j=0}^{d_Y - n_2} (-1)^j \binom{d_Y - n_2}{j} h_{\theta_2}((n_2 + j) h_{\theta_2}^{-1}(p)) \pi(\theta_1, \theta_2) d\theta_1 d\theta_2. \quad (16)
\end{aligned}
$$

And the numerator A is given by

$$
\begin{aligned}
A \; = \; & \sum_{n_1 = n_X}^{d_X} \sum_{n_2 = n_Y}^{d_Y} \lambda_1 \lambda_2 \binom{d_X}{n_1} \binom{d_Y}{n_2} \sum_{k=0}^{d_X - n_1} \sum_{j=0}^{d_Y - n_2} (-1)^{k+j} \binom{d_X - n_1}{k} \binom{d_Y - n_2}{j} \\
& \int_0^{\infty} \int_0^{\infty} h_{\theta_1}((n_1 + k) h_{\theta_1}^{-1}(p)) h_{\theta_2}((n_2 + j) h_{\theta_2}^{-1}(p)) H^{(2)} \left(\lambda_1 \theta_1 + \lambda_2 \theta_2\right) d\theta_1 d\theta_2
\end{aligned}
$$

Conditional probability of default

Evaluating the denominator:

$$
\begin{aligned}
& P\left(N_{\mathbf{Y}}(p)=n_2\right) \\
= & \int_{\Theta_2} P\left(N_{\mathbf{Y}}(p)=n_2 \mid \theta_2\right) \times \pi(\theta_2) d\theta_2 \\
= & -\int_0^{\infty} \binom{d_Y}{n_2} \sum_{j=0}^{d_Y-n_2} (-1)^j \binom{d_Y-n_2}{j} h_{\theta_2}((n_2+j) h_{\theta_2}^{-1}(p)) \lambda_2 H^{(1)}(\lambda_2\theta_2) d\theta_2 \\
= & \lambda_2 \binom{d_Y}{n_2} \sum_{j=0}^{d_Y-n_2} (-1)^{j+1} \binom{d_Y-n_2}{j} \int_0^{\infty} h_{\theta_2}((n_2+j) h_{\theta_2}^{-1}(p)) H^{(1)}(\lambda_2\theta_2) d\theta_2,
\end{aligned}
\tag{17}
$$

- where $\pi(\theta_2) = -\lambda_2 H^{(1)}(\lambda_2\theta_2)$.

Thus, B is given by

$$
B = \sum_{n_2=n_Y}^{d_Y} \lambda_2 \binom{d_Y}{n_2} \sum_{j=0}^{d_Y-n_2} (-1)^{j+1} \binom{d_Y-n_2}{j} \int_0^{\infty} h_{\theta_2}((n_2+j) h_{\theta_2}^{-1}(p)) H^{(1)}(\lambda_2\theta_2) d\theta_2.
$$

Conditional probability of default (limiting case: $p \to 0$)

Deriving limiting behaviour of $P\left(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y\right)$ for $p \to 0$ for the

- Case I: regularly varying (to zero) copula generators $h(x)$ for which it holds for some $\alpha \in \mathbb{R}$:

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = t^{\alpha}, \qquad t > 0.$$

- Case II: rapidly varying (to zero) copula generators if

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = \begin{cases} 0 & \text{for } t > 1, \\ \infty & \text{for } 0 < t < 1. \end{cases}$$

Conditional probability of default (limiting case: $p \to 0$) for the Case 1 (regularly varying)

Theorem 1:

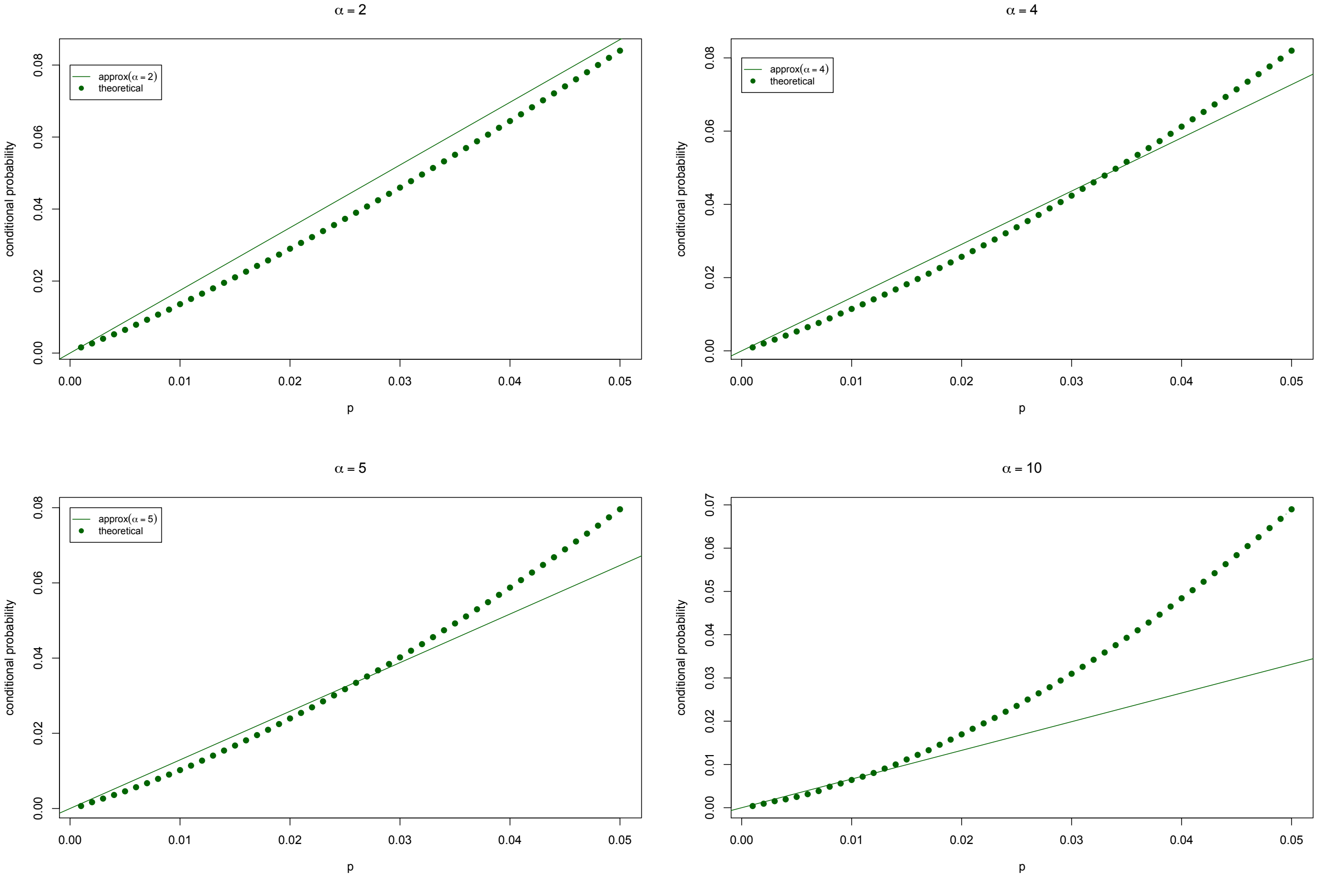
Suppose that the generator $h_\theta(x)$ *is regularly varying at infinity with index* $\alpha = \alpha(\theta)$ *for each* $\theta > 0$. *Then*

$$\lim_{p \to 0} \frac{1}{p} P(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y) = c_1, \tag{18}$$

where the constant c_1 *is given by*

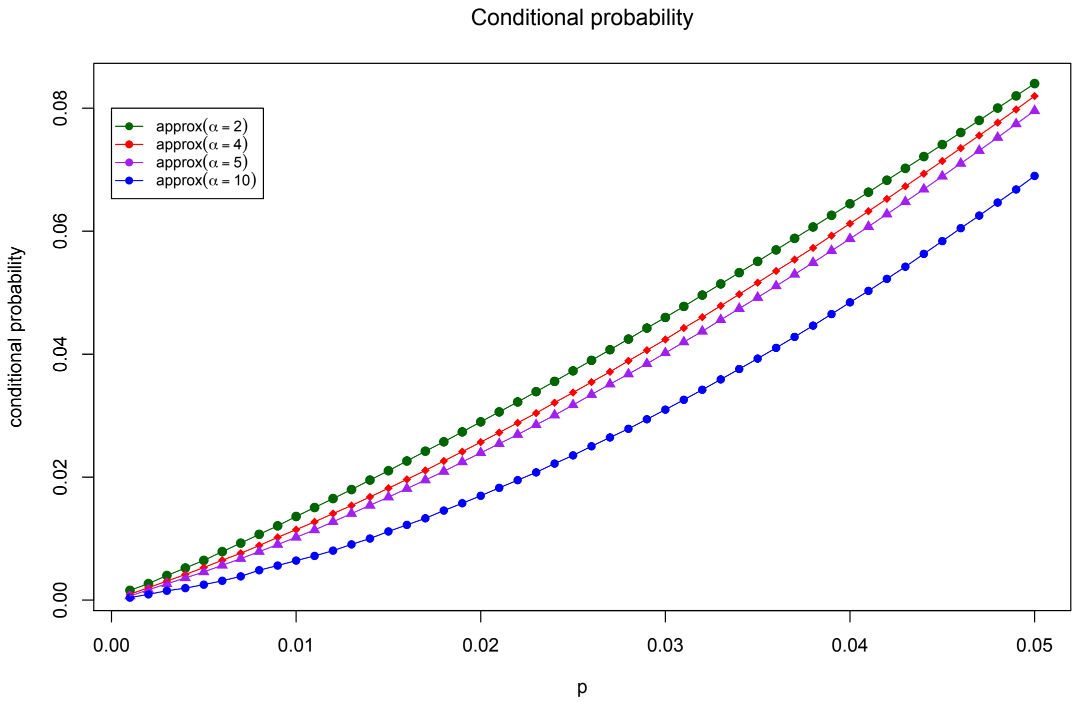
$$\begin{aligned} c_1 \;=\; & \lambda_1 \sum_{n_1=n_X}^{d_X} \sum_{n_2=n_Y}^{d_Y} \binom{d_X}{n_1}\binom{d_Y}{n_2} \sum_{k=0}^{d_X-n_1} \sum_{j=0}^{d_Y-n_2} (-1)^{k+j} \binom{d_X - n_1}{k}\binom{d_Y - n_2}{j} \\ & \int_0^\infty \int_0^\infty (n_1+k)^{\alpha(\theta_1)} (n_2+j)^{\alpha(\theta_2)} H^{(2)} (\lambda_1\theta_1 + \lambda_2\theta_2)\, d\theta_1 d\theta_2 \\ & \div \sum_{n_2=n_Y}^{d_Y} \binom{d_Y}{n_2} \sum_{j=0}^{d_Y-n_2} (-1)^{j+1} \binom{d_Y - n_2}{j} \int_0^\infty (n_2+j)^{\alpha(\theta_2)} H^{(1)}(\lambda_2\theta_2) d\theta_2. \end{aligned}$$

Empirical Results: Clayton-Pareto (Case 1: regularly varying)



Tail probability $P\left(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y\right)$ for the *Clayton-Pareto* case as a function of probability of default p. The results are obtained for the case when h_{θ_1} and h_{θ_2} are generators of *Clayton* copula (intradependence) and a generalized bivariate Pareto distribution is used to model interdependence. Parameters are specified as $d_X = d_Y = 15, n_X = n_Y = 2$, and different values of $\alpha = 2, 4, 5, 10$ controlling strength of interdependence.

Empirical Results: Clayton-Pareto (Case 1: regularly varying)



Approximations for different values of α: Tail probability $P\left(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y\right)$ for the *Clayton-Pareto* case as a function of probability of default p. The results are obtained for the case when h_{θ_1} and h_{θ_2} are generators of *Clayton* copula (intradependence) and a generalized bivariate Pareto distribution is used to model interdependence. Parameters are specified as $d_X = d_Y = 15, n_X = n_Y = 2$, and different values of $\alpha = 2, 4, 5, 10$.

Conditional probability of default (limiting case: $p \to 0$) for the Case 2 (rapidly varying)

Theorem 2:

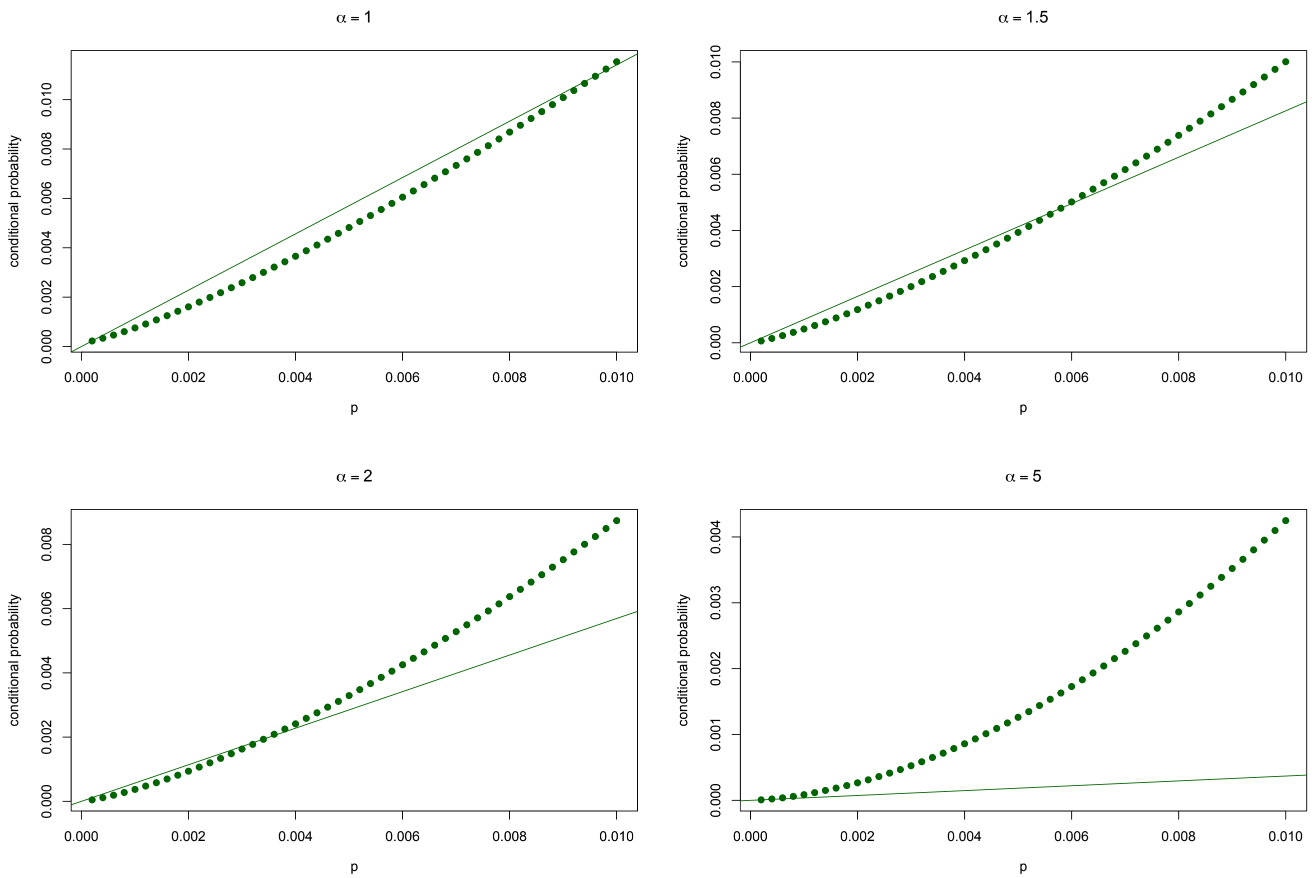
For the Gumbel copula with the generator $h_\theta(x) = \exp\left\{-x^{1/(\theta+1)}\right\}, \theta > 0$, *we have*

$$\lim_{p\to 0} \frac{1}{p} P\left(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y\right) = c_2, \tag{19}$$

where the constant c_2 *is given by*

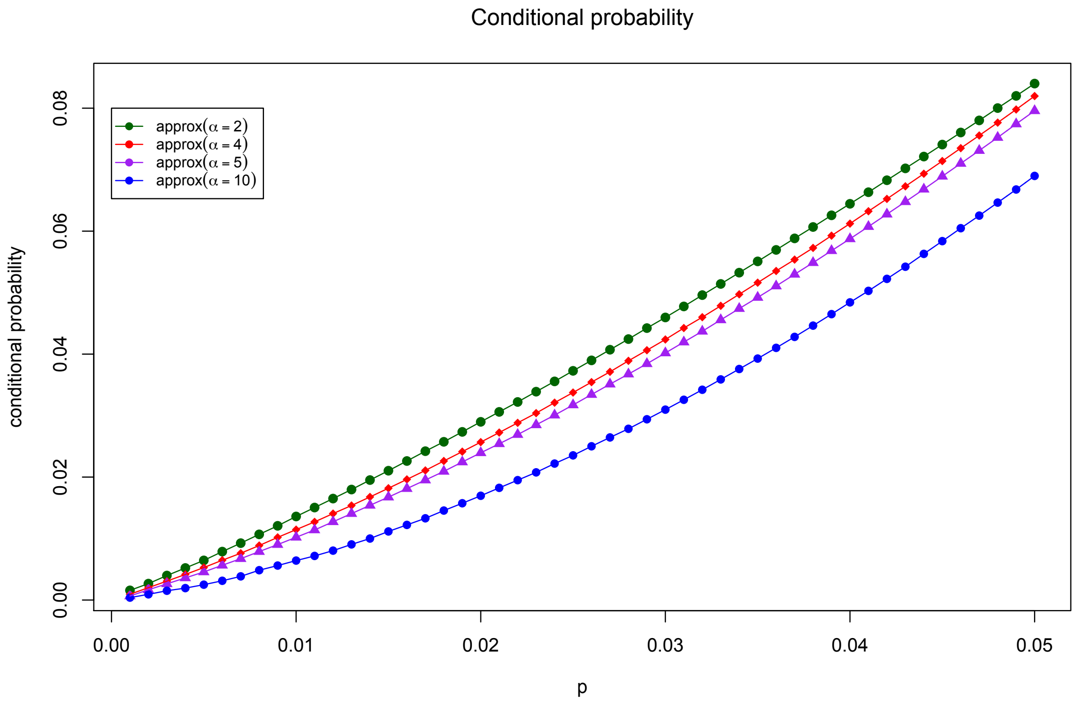
$$\begin{aligned} c_2 &= \lambda_1 \sum_{n_1=n_X}^{d_X} \sum_{n_2=n_Y}^{d_Y} \binom{d_X}{n_1}\binom{d_Y}{n_2} \\ &\quad \sum_{k=0}^{d_X-n_1} \sum_{j=0}^{d_Y-n_2} (-1)^{k+j} \binom{d_X-n_1}{k}\binom{d_Y-n_2}{j} c(n_1+k, n_2+j) \\ &\quad \div \sum_{n_2=n_Y}^{d_Y} \binom{d_Y}{n_2} \sum_{j=0}^{d_Y-n_2} (-1)^j \binom{d_Y-n_2}{j} \frac{\Gamma(\alpha+1)}{\lambda_2^{\alpha+1}(\log(n_2+j))^\alpha} \end{aligned}$$

Empirical Results: Gumbel-Pareto (Case 2: rapidly varying)



Tail probability $P\left(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y\right)$ for the *Gumbel-Pareto* case as a function of probability of default p. The results are obtained for the case when h_{θ_1} and h_{θ_2} are generators of *Gumbel* copula (intradependence) and a generalized bivariate Pareto distribution is used to model interdependence. Parameters are specified as $d_X = d_Y = 15, n_X = n_Y = 2$, and different values of $\alpha = 1, 1.5, 2, 5$ controlling strength of interdependence. In panels 1-4 actual conditional probabilities are shown using solid line and approximations are shown using dotted lines.

Empirical Results: Gumbel-Pareto (Case 2: rapidly varying)



Approximations for different values of α: Tail probability $P\left(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y\right)$ for the *Gumbel-Pareto* case as a function of probability of default p. The results are obtained for the case when h_{θ_1} and h_{θ_2} are generators of *Clayton* copula (intradependence) and a generalized bivariate Pareto distribution is used to model interdependence. Parameters are specified as $d_X = d_Y = 15, n_X = n_Y = 2$, and different values of $\alpha = 1, 1.5, 2, 5$.

Conclusion:

- Analyze the role of financial institutions as potential sources of instability or default
- Investigate default risk propagation within a multilayer system
- The proposed framework has a wide range of applications across financial, insurance, reinsurance, climate and other sectors of the economy.
- Develop a multilayer system featuring the dependence structure between the layers of the financial system (interdependence) as well as within each layer (intradependence).
  - Archimedean copulas used to capture the dependency between the number of losses $N_{\mathbf{X}}(p)$ and $N_{\mathbf{Y}}(p)$, through the parameters θ_1 and θ_2
  - Generalised bivariate Pareto distribution is used to model the dependency between (θ_1, θ_2)
  - This work is a first attempt to tackle propagation effect across the two layers, accomplished in terms of the numbers of defaults.

Conclusion:

- Important limiting result for the conditional probability (for regularly varying and some rapidly varying):
  - Important for assessing default events in practice:

$$\lim_{p \to 0} \frac{1}{p} P\left(N_{\mathbf{X}}(p) \geq n_X | N_{\mathbf{Y}}(p) \geq n_Y\right) = c$$

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Thank You for your attention!

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