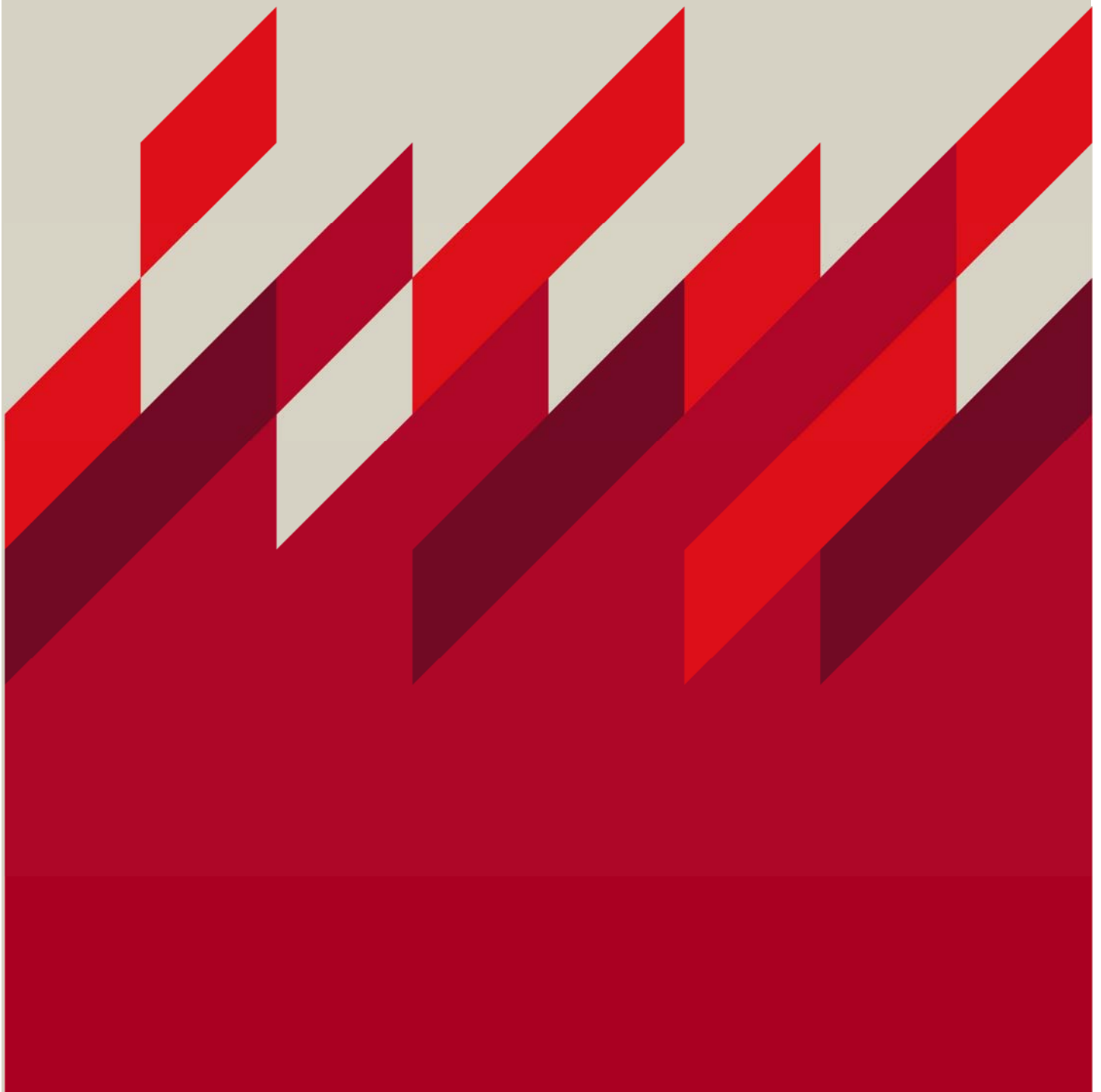




Optimal annuitisation, housing decisions and means-tested public pension in retirement under expected utility stochastic control framework

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Abstract

In this paper we develop a retirement model under the expected utility stochastic control framework to find optimal decisions with respect to consumption, risky asset allocation, access to annuities, reverse mortgage and the option to scale housing. The model is solved numerically using Least-Squares Monte Carlo method adapted to handle expected utility stochastic control problem in high dimensions. To demonstrate the applicability of the framework, the model is applied in the context of the Australian retirement system. Few retirees in Australia utilise financial products in retirement, such as annuities or reverse mortgages. Since the government-provided means-tested Age Pension in Australia is an indirect annuity stream which typically is higher than the average consumption floor, it is argued that this is the reason why Australians do not annuitise. In addition, in Australia where assets allocated to the family home are not included in the means tests of Age Pension, the incentive to over allocate wealth into housing assets is high. This raises the question whether a retiree is really better off over allocating into the family home, while accessing home equity later on either via downsizing housing or by taking out a reverse mortgage. Our findings confirm that means-tested pension crowds out voluntary annuitisation in retirement, and that annuitisation is optimal sooner rather than later once retired. We find that it is never optimal to downscale housing with the means-tested Age Pension when a reverse mortgage is available, only when there is no other way to access equity then downsizing is the only option.

Keywords: Dynamic programming, stochastic control, optimal policy, retirement, means-tested age pension, defined contribution pension

JEL classification: D14 (Household Saving; Personal Finance), D91 (Intertemporal Household Choice; Life Cycle Models and Saving), G11 (Portfolio Choice; Investment Decisions), C61 (Optimization Techniques; Programming Models; Dynamic Analysis)

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1 Introduction

Modelling the retirement phase using life cycle models is a complex task in many aspects. Retirees have many different options for managing and spending their life savings. Most life cycle models offer very limited choices, mainly due to the difficulties and computational limitations of solving such models. While there is a plethora of research on life cycle models in retirement (Emms (2012), Blake et al. (2014), Andréasson et al. (2017), Kingston and Thorp (2005) to name a few), the majority of them only allow very few control, state or stochastic variables, thus limiting the practical applicability of their models. In this paper we develop a retirement model which is based on the basic model in Andréasson et al. (2017) and Ding et al. (2014), extended with a stochastic interest rate, availability of deposit account in addition to a pension account, and control variables for lifetime annuities, reverse mortgages and the option to scale housing. These features make the model more applicable to real life. We develop the Least-Squares Monte Carlo (LSMC) method by utilising the method improvements from Andréasson and Shevchenko (2017b) to solve this high-dimensional stochastic control problem. The model can be adapted to retirement phase in various countries that would require a good knowledge of country specific retirement systems. In this paper we apply the model in the context of the Australian system.

The government-provided Age Pension in Australia is means-tested to provide support for retirees with low wealth and/or income. The means tests raise a number of questions regarding optimal behaviour, such as optimal behaviour with respect to current or planned policies, but also regarding the validity of traditional knowledge in retirement modelling. One such insight is the ‘fact’ that a risk averse retiree tends to be better off by annuitising part of his/her wealth (Yaari, 1965; Davidoff et al., 2005; Milevsky and Young, 2007). A lifetime annuity is a financial product that pays a guaranteed income and insures against outliving one’s assets (longevity risk). By purchasing an annuity the retiree gives up wealth that could potentially earn a higher return and which could be used as bequest. Even after the mortality credit¹, the payout rate is generally low, but insures the retirees from outliving their incomes. Risk averse agents², however, discount the risk premium more and value a protected income over potentially higher future consumption, thus annuitising more wealth (Iskhakov et al., 2015). There are alternative annuities that address the negative aspects of a lifetime annuity, such as variable annuities with guaranteed minimum withdrawal and guaranteed minimum death benefits, which allow for equity growth and bequest motives respectively (Luo and Shevchenko, 2015; Shevchenko and Luo, 2016). These products tend to be more expensive due to the additional benefits. The retiree therefore needs to find a balance between a guaranteed consumption and the possibility to leave a bequest. Yaari (1965) showed that if no bequest motive is present, then full annuitisation is optimal. If such a bequest motive exists, however, annuitisation is still optimal but typically only partial (Davidoff et al., 2005; Friedman and Warshawsky, 1990), which is also the case when a certain consumption floor is present. Despite this, very few Australians annuitise any wealth (Iskhakov et al., 2015; Kingston and Thorp, 2005), which is consistent with retirees globally who receive

¹Mortality credit refers to the discounting of future income streams based on survival probabilities. The value of the future income stream is weighted by the probability of being alive to receive this future income.

²This is true for rational investors only. Irrational investors, however, may value their current level of consumption too much and therefore defer annuitisation (Marín-Solano and Navas, 2010).

other stable income streams (Inkmann et al., 2011; Dushi and Webb, 2004; Kingston and Thorp, 2005). The exception is Switzerland, where the majority of retirees do annuitise (Avanzi, 2010; Avanzi and Purcal, 2014). As the means-tested Age Pension provides an income stream exceeding the consumption floor, the Age Pension becomes a possible substitute for voluntary annuitisation. We therefore examine the optimal level of annuitisation in relation to wealth over time, which in turn relates to the means tests.

Another important aspect of the means tests is the lenient treatment of the family home. Most Australian households do not convert housing assets into liquid assets in order to cover expenses in retirement, with the exception of certain events such as the death of a spouse, divorce, or moving to an aged care facility (Olsberg and Winters, 2005; Asher et al., 2017). However, by allocating more assets to the family home, the means-tested assets can be lowered which in turn results in more Age Pension received, and home equity can be accessed later in retirement if needed. As with annuities, this raises the question whether retirees should access home equity, either by selling the home or through home equity products, or if the means tests crowd out such products as well. Sun et al. (2008) find that the reverse mortgage is a very risky asset, owing to the uncertainty of interest rates and housing markets. However, the decision to access home equity cannot be made purely for financial reasons and needs to be set in the context of typical Australian retirement behaviour. Due to both financial benefits and attachment to their home, and especially neighbourhood, retirees tend to stay homeowners late in life (Olsberg and Winters, 2005). The possibility to borrow money decreases with age, mainly due to having no labour income, and the retiree becomes increasingly locked into their home equity (Nakajima, 2017). An increasingly popular solution is therefore a reverse mortgage, which allows the retiree to borrow against home equity, up to a certain loan-to-value ratio (LVR) threshold. The LVR threshold tends to increase with age. The initial principal limit generally starts with 20-25% at age 65 (subject to expected interest rate and property value), which translates to either the lump sum or the present value of future payments, and increases 1% per year. The house equity is used as collateral and allows the retiree to access housing equity while maintaining residence in the house. The retiree can typically choose between six repayment options: lump sum, line of credit (allowing flexible amounts and payment times), tenure (equal monthly payments), term (tenure but with a fixed time horizon) and combinations of line of credit with either tenure or term (Chen et al., 2010). The loan is charged with either fixed or variable interest, but instead of requiring amortisation or interest payments they accumulate (although the retiree is free to make repayments at any time to reduce debt). The main benefits of such an arrangement are that it limits the risk as the loan repayments are capped at the house value, and allows the retiree to access more equity with age (contrary to traditional loans). However, interest rates are higher due to lending margins and insurance. Chiang and Tsai (2016) find that the desire for reverse mortgages is negatively correlated with the costs (application costs and insurance/spread added to the interest rate) as well as the income for a retiree, and according to Nakajima (2017) the loans are very expensive for retirees. In addition, if a lump sum is received and allocated to what is considered an asset in the means tests, such as a risky investment or simply a bank account, it will affect the Age Pension. On the other hand, if the funds are spent right away they will not have an impact on the Age Pension received.

Previous research found that the Age Pension crowds out decisions that otherwise are optimal (Iskhakov et al., 2015; Büttler et al., 2016). In our paper we evaluate whether such findings are consistent in a more realistic framework. Asher et al. (2017) finds

evidence that few households use financial products to access home equity, such as reverse mortgages. For these reasons, we investigate whether the retiree is better off based on two additional control variables: borrowing against housing assets with a reverse mortgage, and up/downsizing the housing in retirement. Since family home is exempt from the means tests, it might be optimal to over allocate in housing and then draw it down by reverse mortgage. Extending the model with more flexible decisions for homeowners is highly topical: in Australian Government (2017), the government announced that retirees will be able to deposit non-concessional contributions from the proceeds of selling their home into their pension fund account (subject to additional conditions being met). The deposit is capped at \$300,000 per retiree, hence couples can deposit twice the amount. The reason is to encourage house downsizing in retirement, where the additional living space is no longer needed. As these rules will be in effect from the 1st of July 2018 they are not explicitly modelled in this paper, but signifies the importance of understanding the effect that house equity related decisions has on the retiree.

The paper is structured as follows. First, the ‘benchmark’ stochastic model is defined in Section 2 which is the foundation used in this paper. In Section 3, additional optimal controls with respect to annuitisation decisions and home equity access are modelled individually. The results of each extended model are evaluated in Section 4. Finally, the paper is concluded in Section 5.

2 Benchmark model

We utilise the basic model developed in Andréasson et al. (2017), with the same utility functions and parameters, but extend the model in several important aspects³. First, a stochastic risk-free rate is introduced, which is important due to long time horizon of the retirement phase. Second, a deposit account is now available in addition to the pension account, which is important since the pension account does not allow for deposits in retirement. Although the definition of a deposit account normally is that it only pays interest and has restrictions on withdrawals, we use this in a wider sense that allows financial investments, interest rate investments and yearly withdrawals and deposits with no restrictions on size as long as the account balance is non-negative. Later, in Sections 3.1 and 3.2, the model is extended to cover more flexible housing decisions and annuitisation.

2.1 Model

The objective of the retiree is to maximise expected utility generated from consumption, housing and bequest. The retiree of the age t years starts off with a total wealth W , at the age of retirement $t = t_0$ years, and is given the option to allocate a proportion ϱ of wealth into housing $H_{t_0} = \varrho W$ (if he/she is already a homeowner, he/she has the option to adjust current allocation by up- or downsizing). The remaining (liquid) wealth $W_{t_0} = W(1 - \varrho)$ is placed in an Allocated Pension account, which is a special type of account that does not have a tax on investment earnings and is subject to the regulatory minimum withdrawal

³It should be noted that since the model was calibrated to data based on certain assumptions of deterministic variables, changing these to stochastic might have implications on the utility parameters. Using the same parameters does, however, function as a benchmark to evaluate the benefits of additional decisions and extensions to the model. We do not say in this case that the average Australian retiree is recommended to act based on the model solution, only that such a solution can show whether the retiree is better or worse off with regards to the decisions.

rates (current rates are provided in Table 3) that depend on the age of the retiree. In addition to the pension account, the retiree has access to a deposit account \widetilde{W}_t which is an account that holds liquid wealth separate from the pension account, which is taxable and where the balance is included in the means tests. The purpose of such an account is that the retiree will be able to save part of the Age Pension and/or drawdowns from the pension account when minimum withdrawals are larger than what is optimal to consume. Such an account is necessary later on in Sections 3.1 and 3.2, as it is possible to receive lump sums but pension accounts do not allow funds to be added to them after retirement.

We consider couple and single retiree households (the Age Pension treats couples as a single entity) where possible states of the household are modelled by a family-status random variable

$$G_t \in \mathcal{G} = \{\Delta, 0, 1, 2\}, \quad (1)$$

where Δ corresponds to the household already deceased at time $t - 1$, 0 corresponds to the household deceased during $(t - 1, t]$, and 1 and 2 correspond to the household being alive at time t in a single or couple state respectively. Evolution in time of the family state variable G_t is subject to survival probabilities. In the case of a couple household, there is a risk each time period that one of the spouses passes away, in which case, it is treated as a single household model for the remaining years.

At the start of each year $t = t_0, t_0 + 1, \dots, T - 1$, the retiree will receive a means-tested Age Pension P_t and will decide what amount of saved liquid wealth from the pension account W_t and deposit account \widetilde{W}_t will be used for consumption (defined as proportion drawdown α_t of liquid wealth). Here, T is the maximum age of the agent beyond which survival is deemed impossible. Consumption each period equals received Age Pension and drawdowns:

$$C_t = \alpha_t(W_t + \widetilde{W}_t) + P_t. \quad (2)$$

It can be argued whether a second control variable for consumption from the deposit account is required, as the retiree now can choose what account to withdraw from (as long as the regulatory minimum withdrawal requirement in the pension account is satisfied). However, it is assumed the retiree first draws wealth from the pension account up to the minimum withdrawal rate ν_t each period, and in case optimal consumption exceeds this amount the difference is taken from the deposit account (as long as sufficient funds are available in the deposit account). Due to the deposit account attracting a tax on earnings while the Allocated Pension account is tax-free, it is always optimal to deplete the deposit account first. Hence a separate control variable for withdrawal from deposit account is not required.

Any remaining liquid wealth after drawdown can be invested in a risky asset with real⁴ stochastic annual return Z_t and a cash asset growing at continuous interest rate r_t , where δ_t determines the proportion invested in the risky asset. The stochastic returns of the risky asset Z_t are modelled as independent and identically distributed random variables from a normal distribution $\mathcal{N}(\mu, \sigma_Z^2)$ with mean μ defined in real terms and variance σ_Z^2 . The stochastic short rate r_t is modelled as a Vasicek process

$$dr_t = b(\bar{r} - r_t)dt + \sigma_R dB(t), \quad (3)$$

where $b > 0$ is the speed of reversion to the mean, \bar{r} is the mean the process reverts to, $\sigma_R > 0$ is the volatility and $B(t)$ the standard Brownian motion. The corresponding

⁴By defining the model in real terms (adjusted for inflation), time-dependent variables do not have to include inflation, which otherwise would be an additional stochastic variable.

process discretised in time is

$$r_{t+1} = \bar{r} + e^{-b\Delta_t}(r_t - \bar{r}) + \sqrt{\frac{\sigma_R^2}{2b}(1 - e^{-2b\Delta_t})}\epsilon_{t+1}, \quad (4)$$

where Δ_t is the time between t and $t+1$, and $\epsilon_t \sim \mathcal{N}(0, 1)$ are i.i.d. random disturbances from standard Normal distribution. The Vasicek process is chosen as it allows for negative interest rates, which is suitable as the rate is defined in real terms. A negative interest rate would then indicate that inflation is higher than the nominal risk-free rate. The accumulation of interest over one year is denoted as

$$\tilde{r}_{t,t+1} = \int_t^{t+1} r_u du,$$

and the distribution of $\tilde{r}_{t,t+1}$ can be found in closed form. We could also assume that the cash account grows at the annual deposit rate derived from one-year bond prices, or approximate $\tilde{r}_{t,t+1}$ by r_t , but it does not lead to a material difference in the results. Other one-factor stochastic interest rate models can also be considered, but the Vasicek model is good enough for the purposes of this paper.

It is assumed that the deposit account is invested in the same way as the Allocated Pension account, but the deposit account must pay taxes on any earnings. Transitions for the pension account and the deposit account depend on whether the deposit account can cover any desired drawdowns above the minimum withdrawal rates ν_t . Thus, if the deposit account is large enough to cover any consumption above the minimum withdrawal from pension account, $\widetilde{W}_t(1 - \alpha_t) > W_t(\alpha_t - \nu_t)$, the evolution for the pension account is

$$W_{t+1} = W_t(1 - \nu_t) \times (\delta_t e^{Z_{t+1}} + (1 - \delta_t)e^{\tilde{r}_{t,t+1}}). \quad (5)$$

For the deposit account, the evolution is

$$\begin{aligned} \widetilde{W}_{t+1} = & [\widetilde{W}_t(1 - \alpha_t) + W_t(\nu_t - \alpha_t)] \times (\delta_t e^{Z_{t+1}} + (1 - \delta_t)e^{\tilde{r}_{t,t+1}}) \\ & - \Theta(\widetilde{W}_t(1 - \alpha_t) + W_t(\nu_t - \alpha_t), \delta_t e^{Z_{t+1}} + (1 - \delta_t)e^{\tilde{r}_{t,t+1}}), \end{aligned} \quad (6)$$

where the function Θ calculates the tax on asset growth, and is defined as

$$\Theta(w, z) = 0.15w \max(z - 1, 0). \quad (7)$$

Note that only the deposit account attracts a tax on earnings. For simplicity, it is assumed that any gains are realised each year, and that the tax rate is 15%⁵. If consumption is less than minimum withdrawals, the excess funds are stored in the deposit account. On the other hand, if $\widetilde{W}_t(1 - \alpha_t) \leq W_t(\alpha_t - \nu_t)$, the deposit account is depleted to zero (and thus $\widetilde{W}_{t+1} = 0$) and the excess consumption comes from the pension account which evolves as

$$W_{t+1} = (W_t + \widetilde{W}_t)(1 - \alpha_t) \times (\delta_t e^{Z_{t+1}} + (1 - \delta_t)e^{\tilde{r}_{t,t+1}}). \quad (8)$$

Denote the vector of state variables as $X_t = (W_t, \widetilde{W}_t, G_t, H_t, r_t)$. Each period the agent receives utility based on the current state of family status G_t :

$$R_t(X_t, \alpha_t) = \begin{cases} U_C(C_t, G_t, t) + U_H(H_t, G_t), & \text{if } G_t = 1, 2, \\ U_B(W_t + \widetilde{W}_t + H_t), & \text{if } G_t = 0, \\ 0, & \text{if } G_t = \Delta. \end{cases} \quad (9)$$

⁵Due to the many tax offsets, rebates and investment options in retirement which can alter the effective tax rate, the tax rate has been set to a fixed 15% which equals the earnings tax on Self-Managed Super Funds.

That is, if the agent is alive, he/she receives reward (utility) based on consumption U_C and housing U_H ; if he/she died during the year $(t - 1, t]$, the reward comes from the bequest U_B ; and if he/she died before or at $t - 1$, there is no reward. Note that the reward received when the agent is alive depends on whether the family state is a couple or single household due to different utility parameters and Age Pension thresholds.

Finally, at $t = T$ the terminal reward function is given as:

$$\tilde{R}(X_T) = \begin{cases} U_B(W_T + \tilde{W}_T, H_T), & \text{if } G_T \geq 0, \\ 0, & \text{if } G_T = \Delta. \end{cases} \quad (10)$$

We use the same definition of consumption, bequest and housing utility functions as in Andréasson et al. (2017), where parameterization and interpretation are discussed in detail.

- Consumption utility function:

$$U_C(C_t, G_t, t) = \frac{1}{\psi^{t-t_0} \gamma_d} \left(\frac{C_t - \bar{c}_d}{\zeta_d} \right)^{\gamma_d}, \quad d = \begin{cases} C, & \text{if } G_t = 2 \quad (\text{couple}), \\ S, & \text{if } G_t = 1 \quad (\text{single}), \end{cases} \quad (11)$$

where $\gamma_d \in (-\infty, 0)$ denotes the risk aversion, \bar{c}_d is the consumption floor, ζ_d is a scaling factor that normalises utility of couple and single households. These parameters are subject to family state G_t . Finally, $\psi \in [1, \infty)$ is a “health” proxy to control for decreasing consumption with age.

- Bequest utility function:

$$U_B(W_t + \tilde{W}_t + H_t) = \left(\frac{\theta}{1 - \theta} \right)^{1 - \gamma_S} \frac{\left(\frac{\theta}{1 - \theta} a + W_t + \tilde{W}_t + H_t \right)^{\gamma_S}}{\gamma_S}, \quad (12)$$

where W_t denotes the liquid assets available for bequest, γ_S denotes the risk aversion parameters of a singles household, $\theta \in [0, 1)$ the utility parameter for bequest preferences over consumption, and $a \in \mathbb{R}^+$ the threshold for luxury bequest.

- Housing utility function:

$$U_H(H_t, G_t) = \begin{cases} \frac{1}{\gamma_H} \left(\frac{\lambda_d H_t}{\zeta_d} \right)^{\gamma_H}, & \text{if } H_t > 0, \\ 0, & \text{if } H_t = 0, \end{cases} \quad (13)$$

where γ_H is the risk aversion parameter for housing (different from risk aversion for consumption and bequest), H_t is the value of the family home and $\lambda_d \in (0, 1]$ is the housing preference defined as a proportion of the market value. In the benchmark model H_t is assumed to be constant (in real terms) for all t .

The retiree has to find the decisions that maximise the total expected utility with respect to the consumption, investment and housing. This is defined as a stochastic control problem, where decisions (controls) at time t depend on the realisation of stochastic state variables W_t, \tilde{W}_t, r_t and G_t at time t with unknown future realisations. Then, the overall problem of maximization of expected utility is defined as:

$$\max_{\varrho} \left[\sup_{\alpha, \delta} \mathbb{E}_{t_0}^{\alpha, \delta} \left[\beta_{t_0, T} \tilde{R}(X_T) + \sum_{t=t_0}^{T-1} \beta_{t_0, t} R_t(X_t, \alpha_t) \mid X_{t_0} \right] \right], \quad (14)$$

where $\mathbb{E}_{t_0}^{\alpha, \delta}[\cdot]$ is the expectation with respect to the state vector X_t for $t = t_0 + 1, \dots, T$, conditional on the state variables at time $t = t_0$ if we use controls $\alpha = (\alpha_{t_0}, \alpha_{t_0+1}, \dots, \alpha_{T-1})$ and $\delta = (\delta_{t_0}, \delta_{t_0+1}, \dots, \delta_{T-1})$ for $t = t_0, t_0 + 1, \dots, T - 1$. The subjective discount rate $\beta_{t,t'}$ is a proxy for personal impatience between time t and t' . This problem can be solved numerically with dynamic programming using backward in time recursion of the Bellman equation

$$V_t(X_t) = \sup_{\alpha_t, \delta_t} \left\{ R_t(X_t, \alpha_t) + \mathbb{E}_t^{\alpha_t, \delta_t} [\beta_{t,t+1} V_{t+1}(X_{t+1}) \mid X_t] \right\}, \quad (15)$$

for $t = T - 1, \dots, 0$, starting from the terminal condition

$$V_T(X_T) = \tilde{R}(X_T). \quad (16)$$

Then, optimal housing decision control ρ maximising $V_0(X_0)$ is calculated. Note that the death probabilities are not explicit in the objective function, but affect the evolution of the family status and, thus, are involved in the calculation of the conditional expectation. Later in Section 3.2 we will also consider housing decisions over time.

2.1.1 Age Pension

In Australia, retirees aged 65.5⁶ are entitled to Age Pension and can receive at most the full Age Pension, which decreases as assets and/or income increase and is determined by the income and asset tests. Income streams from Allocated Pension accounts⁷ and financial assets are based on deemed income, which refers to a progressive assumed return from financial assets without reference to the actual returns on the assets held. Therefore, the income test can depend on both labour income (if any), deemed income from financial investments not held in the Allocated Pension account and deemed income on Allocated Pension accounts. Two different deeming rates may apply based on the value of the account: a lower rate ζ_- for assets under the deeming threshold κ_d and a higher rate ζ_+ for assets exceeding the threshold.

The Age Pension received is modelled with respect to the current liquid assets, where the combined account values of the deposit and pension account are used for the asset test. The Age Pension function can be defined as

$$P_t := f(W_t + \tilde{W}) = \max [0, \min [P_{\max}^d, \min [P_A, P_I]]], \quad (17)$$

where P_{\max}^d is the full Age Pension, P_A is the asset test and P_I is the income test functions. The P_A function can be written as

$$P_A := P_{\max}^d - (W_t + \tilde{W} - L_A^{d,h}) \varpi_A^d, \quad (18)$$

where $L_A^{d,h}$ is the threshold for the asset test and ϖ_A^d the taper rate for assets exceeding the thresholds. Superscript d is the categorical index indicating couple or single household status as defined in Equation (11). The variables are subject to whether it is a single or couple household, and the threshold for the asset test is also subject to whether the

⁶This is the current age as of July 2017, which will be increased six months every two years until reaching 67 at 2023 (<https://www.humanservices.gov.au/individuals/enablers/age-rules-age-pension>).

⁷This applies to Allocated Pension accounts opened after 1 January 2015 (<http://guides.dss.gov.au/guide-social-security-law/3/9/3/31>). Older accounts may have different rules which are not considered in this paper

household is a homeowner or not ($h = \{0, 1\}$). The function for the income test can be written as

$$P_I := P_{\max}^d - (P_D(W_t + \widetilde{W}) - L_I^d)\varpi_I^d, \quad (19)$$

$$P_D(W_t + \widetilde{W}) = \varsigma_- \min [W_t + \widetilde{W}, \kappa^d] + \varsigma_+ \max [0, W_t + \widetilde{W} - \kappa^d], \quad (20)$$

where L_I^d is the threshold for the income test and ϖ_I^d the taper rate for income exceeding the threshold. Function $P_D(W_t + \widetilde{W})$ calculates the deemed income, where κ^d is the deeming threshold, and ς_- and ς_+ are the deeming rates that apply to assets below and above the deeming threshold, respectively. The parameters for the current Age Pension policy are presented in Table 2.

2.2 Stochastic control problem definition

For the purpose of a complete definition of the benchmark model, it is defined in the stochastic control problem framework.

- Denote a state vector as $X_t = (W_t, \widetilde{W}_t, G_t, H_t, r_t) \in \mathcal{W} \times \mathcal{W} \times \mathcal{G} \times \mathcal{H} \times \mathcal{R}$, where $W_t \in \mathcal{W} = \mathbb{R}^+$ and $\widetilde{W}_t \in \mathcal{W} = \mathbb{R}^+$ denotes the current level of liquid wealth in a pension account and a deposit account respectively. $G_t \in \mathcal{G} = \{\Delta, 0, 1, 2\}$ denotes whether the agent is dead, died in this period, is alive in a single household, or is alive in a couple household. The stages are sequential; hence, an agent that starts out as a couple becomes single when one spouse dies. $H_t \in \mathcal{H} = \mathbb{R}^+$ denotes the value of the home and $r_t \in \mathcal{R} = \mathbb{R}$ the stochastic real risk-free interest rate (thus can take on negative values).
- Denote an action space of $(\alpha_t, \delta_t, \varrho) \in \mathcal{A} = (-\infty, 1] \times [0, 1] \times \{0, [\frac{H_L}{W}, 1]\}$ for $t = t_0$, and $(\alpha_t, \delta_t) \in \mathcal{A} = (-\infty, 1] \times [0, 1]$ for $t = t_0 + 1, \dots, T - 1$. Here, $\varrho \in \{0, [\frac{H_L}{W}, 1]\}$ is the proportion total wealth allocated to housing, $\alpha_t \in (-\infty, 1]$ denotes the proportion of wealth consumed and $\delta_t \in [0, 1]$ is the percentage of wealth allocated in the risky asset. The upper boundary of 1 indicates that the drawdown cannot be larger than the total wealth, nor invest more than 100% in risky assets; hence, borrowing is not allowed. However, negative values for drawdown are allowed as they represent savings from Age Pension payments into the deposit account. Housing requires a certain minimum down payment H_L , and cannot exceed total wealth at retirement.
- Denote an admissible space of state-action combination as $D_t(x_t) = \{\pi_t(x_t) \in \mathcal{A} \mid \alpha_t \geq \frac{\bar{c}_d - P_t}{W_t + \widetilde{W}_t}\}$, where \bar{c}_d is a consumption floor subject to family stats $d = S$ (single) or $d = C$ (couple). The admissible space includes the possible actions for the current state and indicates that withdrawals must be sufficiently large to cover the necessary consumption floor.
- There exist transition functions for the state variables W_t , \widetilde{W}_t and r_t . As housing is constant in retirement, $H_{t+1} = H_t$. Define the total transition function

$$T_t(W_t, \widetilde{W}_t, r_t, \alpha_t, \delta_t, z_{t+1}) = \begin{bmatrix} T_t^W(W_t, \alpha_t, \delta_t, z_{t+1}, \tilde{r}_{t,t+1}) \\ T_t^{\widetilde{W}}(\widetilde{W}_t, \alpha_t, \delta_t, z_{t+1}, \tilde{r}_{t,t+1}) \\ T_t^r(r_t) \end{bmatrix}. \quad (21)$$

Here, $T_t^W(\cdot)$ is the transition function for the pension account

$$T_t^W(\cdot) := W_{t+1} = \begin{cases} W_t(1 - \alpha_t) \\ \times (\delta_t e^{z_{t+1}} + (1 - \delta_t) e^{\tilde{r}_{t,t+1}}), & \text{if } \widetilde{W}_t(1 - \alpha_t) > W_t(\alpha_t - \nu_t), \\ (W_t(\nu_t - \alpha_t) + \widetilde{W}_t(1 - \alpha_t)) \\ \times (\delta_t e^{z_{t+1}} + (1 - \delta_t) e^{\tilde{r}_{t,t+1}}), & \text{otherwise,} \end{cases} \quad (22)$$

where z_{t+1} and $\tilde{r}_{t,t+1}$ are the realisations of the return on the stochastic investment portfolio and accumulated interest on cash assets respectively, over $(t, t + 1]$. We assume that the agent is small and cannot influence the asset price. $T_t^{\widetilde{W}}(\cdot)$ is the transition function for the deposit account

$$T_t^{\widetilde{W}}(\cdot) := \widetilde{W}_{t+1} = \begin{cases} (\widetilde{W}_t(1 - \alpha_t) + W_t(\nu_t - \alpha_t)) \\ \times (\delta_t e^{z_{t+1}} + (1 - \delta_t) e^{\tilde{r}_{t,t+1}}) & \text{if } \widetilde{W}_t(1 - \alpha_t) > \\ - \Theta(\widetilde{W}_t(1 - \alpha_t) + W_t(\nu_t - \alpha_t), & W_t(\alpha_t - \nu_t), \\ \delta_t e^{z_{t+1}} + (1 - \delta_t) e^{\tilde{r}_{t,t+1}}), & \\ 0, & \text{otherwise.} \end{cases} \quad (23)$$

Finally, $T_t^r(\cdot)$ is the transition function for the stochastic interest rate, which is based on equation (4), hence

$$T_t^r(r_t) := r_{t+1} = \bar{r} + e^{-b}(r_t - \bar{r}) + \sqrt{\frac{\sigma_R^2}{2b}(1 - e^{-2b})} \epsilon_{t+1}. \quad (24)$$

- Denote the stochastic transitional kernel as $Q_t(dx'|x, \pi_t(x))$, which represents the probability of reaching a state in $dx' = (dw_{t+1}, d\widetilde{w}_{t+1}, g_{t+1}, dr_{t+1})$ at time $t + 1$ if action $\pi_t(x)$ is applied in state x at time t . The transition probability for W_{t+1} , \widetilde{W}_{t+1} and r_{t+1} are determined by the distributions of $Z_{t+1} \stackrel{i.i.d.}{\sim} \mathcal{N}(\mu, \sigma_Z^2)$, $r_{t+1} \stackrel{i.i.d.}{\sim} \mathcal{N}(\bar{r} + e^{-b}(r_t - \bar{r}), \frac{\sigma_R^2}{2b}(1 - e^{-2b}))$, and $\tilde{r}_{t,t+1} \stackrel{i.i.d.}{\sim} \mathcal{N}(\frac{1}{b}(1 - e^{-b})(r_t - \bar{r}) + \bar{r}, \frac{\sigma_R^2}{b^2}(1 - \frac{2}{b}(1 - e^{-b}) + \frac{1}{2b}(1 - e^{-2b})))$ with $\text{cov}(r_{t+1}, \tilde{r}_{t,t+1}) = \frac{\sigma_R^2}{2b^2}(1 - 2e^{-b} + e^{-2b})$. For simplicity, one can approximate $\tilde{r}_{t,t+1}$ by r_t that will not lead to material difference in the results. As the problem is solved with a simulation based method, the stochastic kernel with respect to the financial stochastic variables does not have to be explicitly defined. The survival probabilities will, however, be implemented directly in the calculations. Let $q(g_{t+1}, g_t)$ denote $\Pr[G_{t+1} = g_{t+1} | G_t = g_t]$. The stochastic kernel is then given by

$$\begin{aligned} & Q_t(dx'|x, \pi_t(x)) \\ &= \Pr[W_{t+1} \in dw_{t+1}, \widetilde{W}_{t+1} \in d\widetilde{w}_{t+1}, G_{t+1} = g_{t+1}, r_{t+1} \in dr_{t+1} | X_t = x_t] \\ &= \Pr[W_{t+1} \in dw_{t+1}, \widetilde{W}_{t+1} \in d\widetilde{w}_{t+1}, r_{t+1} \in dr_{t+1} | W_t = w_t, \widetilde{W}_t = \widetilde{w}_t, r_t] \\ & \quad \times q(g_{t+1}, g_t). \end{aligned} \quad (25)$$

The probabilities for family status are defined as

$$\begin{aligned} q(2, 2) &= p_t^C, & q(1, 2) &= 1 - p_t^C, \\ q(1, 1) &= p_t^S, & q(0, 1) &= 1 - p_t^S, \\ q(\Delta, 0) &= q(\Delta, \Delta) = 1, \end{aligned} \quad (26)$$

where p_t^C is the probability of surviving for one more year as a couple or p_t^S as a single that can be easily estimated from official Life Tables as in Andréasson et al. (2017). All other transition probabilities for family status are 0.

2.3 Parameterisation

The model parameters are taken from Andréasson and Shevchenko (2017a), which were calibrated to Australian empirical retirement data. All utility model parameter values are shown in Table 1. The risky asset annual return is assumed to be from Normal distribution with mean set to 0.056 and variance set to 0.018, which are parameters estimated from S&P ASX/200 data in Andréasson et al. (2017). In addition, we set the terminal age $T = 100$, the minimum down payment for housing $H_L = \$30,000$ and time impatience discounting $\beta = 0.995$. Model parameters not stated here are parameterised in Section 3.1 or 3.2.

Table 1: Model parameters.

	γ_d	γ_H	θ	a	\bar{c}_d	ψ	λ	ζ_d
Single household	-1.98	-1.87	0.96	\$27,200	\$13,284	1.18	0.044	1.0
Couples household	-1.78	-1.87	0.96	\$27,200	\$20,607	1.18	0.044	1.3

The Age Pension parameters are from July 2017 and shown in Table 2, while the minimum withdrawal rates ν_t for Allocated Pension accounts are shown in Table 3. Mortality probabilities are based on unisex data, and taken from Life Tables published by Australian Bureau of Statistics (2014).

Table 2: Age Pension rates published by Centrelink as at June 2017 (www.humanservices.gov.au/customer/services/centrelink/age-pension, accessed June 5, 2017).

		Single	Couple
P_{\max}^d	Full Age Pension per annum	\$22,721	\$34,252
	Income-Test		
L_1^d	Threshold	\$4,264	\$7,592
ϖ_1^d	Rate of Reduction	\$0.5	\$0.5
	Asset-Test		
$L_1^{d,h=1}$	Threshold: Homeowners	\$250,000	\$450,000
$L_1^{d,h=0}$	Threshold: Non-homeowners	\$375,000	\$575,000
ϖ_A^d	Rate of Reduction	\$0.078	\$0.078
	Deeming Income		
κ^d	Deeming Threshold	\$49,200	\$81,600
ς_-	Deeming Rate below κ^d	1.75%	1.75%
ς_+	Deeming Rate above κ^d	3.25%	3.25%

Table 3: Minimum regulatory withdrawal rates for Allocated Pension accounts (<https://www.ato.gov.au/rates/key-superannuation-rates-and-thresholds/?page=10>, accessed June 5, 2017).

Age	≤64	65–74	75–79	80–84	85–89	90–94	≤95
Min. drawdown	4%	5%	6%	7%	9%	11%	14%

3 Extensions

The model is now extended to include: annuitisation (extension 1) and scaling housing/reverse mortgage (extension 2). Note that extension 1 does not apply in extension 2 and vice versa - they are separate and independent extensions which isolate the impact each extension has on optimal control.

3.1 Extension 1 - Annuitisation

The argument why the Australian market has shown such a lack of interest in annuities comes down to the fact that the Age Pension is indirectly an indexed life annuity which pays a known and increasing amount as wealth and income decrease, hence crowding out annuitisation (Iskhakov et al., 2015; Büttler et al., 2016). The Age Pension provides an implicit insurance against both longevity and financial risk, which otherwise is the main argument to annuitise. If annuities were exempt from the means tests, then it would be reasonable to expect an increased interest in annuities. However, the annuity value as well as the annuity payment are included in the means tests. Any annuitisation would therefore give up ‘free’ money if the means tests are binding, as well as give up potential equity growth, unless the annuity is of an equity-linked type.

3.1.1 Annuity pricing

The retiree can each year decide if he/she wants to annuitise any wealth, hence making the annuity indirectly a deferred one by saving wealth in retirement in order to annuitise later (similar to Milevsky and Young (2007)). This introduces the possibility for the retiree to receive additional equity growth on the wealth yet to be annuitised, although with the risk associated, but without requiring more complex annuity products.

Assume an immediate lifetime annuity that is fairly priced (i.e. there are no commercial markups or fees) with constant real payments, where the actuarial present value can be written as

$$a_t(y) := \sum_{i=t+1}^T {}_i p_t J(t, i, y), \quad (27)$$

where $J(t, i, y)$ represents the price of an inflation linked zero coupon bond at time t with maturity i and face value y (the constant real annuity payment, hence adjusted for inflation), ${}_i p_t$ is the probability of surviving from year t to i . The price of this kind of annuity equals a portfolio of mortality risk weighted bonds with maturities from $t + 1$ up to T . At time t , the price of a bond with maturity t' is

$$J(t, t', y) = y \mathbb{E}^{\tilde{Q}}[e^{-\int_t^{t'} r_\tau d\tau}] := y e^{-r(t,t')(t'-t)}, \quad (28)$$

where \tilde{Q} is the risk-neutral measure for pricing interest rate derivatives and $r(t, t')$ is the zero rate (yield) from t to t' . The corresponding Vasicek risk-neutral process is

$$dr_t = [b(\bar{r} - r_t) - \lambda\sigma_R]dt + \sigma_R d\tilde{B}(t), \quad (29)$$

where λ is the market price of risk. The formulas for the bond price and corresponding zero rate can easily be calculated (see, e.g., Hull (2012))

$$r(t, t') = \frac{-\ln A(t, t') + B(t, t')r_t}{t' - t}, \quad (30)$$

where

$$A(t, t') = \exp \left[(B(t, t') - t' + t) \left(\bar{r} - \frac{\lambda\sigma_R}{b} - \frac{\sigma_R^2}{2b^2} \right) - \frac{\sigma_R^2}{4b} B(t, t')^2 \right], \quad (31)$$

$$B(t, t') = \frac{1}{b} \left(1 - e^{-b(t'-t)} \right). \quad (32)$$

Equation (30) gives the full term structure of zero rates of different maturities. In order to fit the real risk-free rate parameters, which are needed to find the correct discounting of the annuity payment, the process outlined in Hull (2012) is used. First the risk-free rate r_t process needs to be parameterised from real data. The Australian cash rate adjusted for inflation is chosen to represent a real risk-free rate which the retiree has access to, where the dataset⁸ contains rates from 1990–2017 in quarterly intervals. Then parameters of the Vasicek model are estimated using Maximum Likelihood method applied to the discretized version of the Vasicek model (equation 4)

$$\max_{b, \bar{r}, \sigma} \sum_{i=1}^n \left(-\frac{1}{2} \ln \left(\frac{\pi\sigma_R^2}{b} (1 - e^{-2b\Delta t}) \right) - \frac{(r_i - \bar{r} - e^{-b\Delta t}(r_{i-1} - \bar{r}))^2}{\frac{\sigma_R^2}{2b}(1 - e^{-2b\Delta t})} \right), \quad (33)$$

where r_i is the observed real cash rate at time t_i . The parameter estimates can be found in closed form and for the considered dataset $\hat{b} = 0.120$, $\hat{\bar{r}} = 0.021$ and $\hat{\sigma}_R = 0.012$. The present real risk-free rate is set to $r_0 = -0.003$, as inflation was higher than the cash rate in the last available data. The market price of risk parameter λ can be estimated by minimising the sum of squared difference between the observed term structure of the zero coupon market rates⁹ and model predicted zero rates (30) over trading dates $t_i, i = 1, \dots, n$ and maturities $T_j, j = 1, \dots, J$:

$$\min_{\lambda} \sum_i \sum_j (r(t_i, t_i + T_j) - r_{i,j}^{obs})^2, \quad (34)$$

where $r_{i,j}^{obs}$ represents the observed yield at date t_i with maturity T_j . The estimate comes out as $\hat{\lambda} = -0.050$, hence the risk-neutral parameter for the mean rate is $\hat{\bar{r}} - \hat{\lambda}\hat{\sigma}_R/\hat{b} = 0.026$, and the other equals the estimates where $b = \hat{b} = 0.120$ and $\sigma_R = \hat{\sigma}_R = 0.012$. The present value of the annuity (27) can now be calculated.

⁸Taken from <https://www.quandl.com/data/RBA/F13-International-Official-Interest-Rates> and <https://tradingeconomics.com/australia/inflation-cpi>

⁹Taken from https://www.quandl.com/data/RBA/F17_0-Zero-Coupon-Interest-Rates-Analytical-Series-Yields

3.1.2 Problem definition

In the context of the life cycle model, the retiree can at any time $t_0, \dots, T-1$ make a (non-reversible) decision $\vartheta_t \in [0, 1 - \alpha_t]$ to annuitise a proportion of liquid wealth ($W_t + \widetilde{W}_t$). As the annuity is of the type annuity-immediate, the retiree makes decision at time t to annuitise and the annuity payments start from $t+1$. The cost (expected future payments) is therefore discounted to t and decreases the wealth immediately in order to protect future consumption. The decision to annuitise leads to a new state variable $Y_{t_0} = 0, Y_t \in \mathcal{Y} = \mathbb{R}^+$, which holds the information of the size of annuity payments each period. The transition function for the state variable is

$$T_t^Y(Y_t, y_t) := Y_{t+1} = Y_t + y_t, \quad (35)$$

where y_t is found from equation (27) by setting the decision to annuitise equal to the actuarial present value, $a_t(y_t) = \vartheta_t(W_t + \widetilde{W}_t)$, and solving for y_t . Note that y_t will always be non-zero due to the non-reversibility of the annuitisation decision. The transition functions for W_t and \widetilde{W}_t are updated to

$$T_t^W(\cdot) := \begin{cases} \begin{aligned} &W_t(1 - \alpha_t - \vartheta_t) \\ &\times (\delta_t e^{z_{t+1}} + (1 - \delta_t)e^{\tilde{r}_{t,t+1}}), \end{aligned} & \text{if } \widetilde{W}_t(1 - \alpha_t - \vartheta_t) > W_t(\alpha_t + \vartheta_t - \nu_t), \\ \begin{aligned} &(W_t + \widetilde{W}_t)(1 - \alpha_t - \vartheta_t) \\ &\times (\delta_t e^{z_{t+1}} + (1 - \delta_t)e^{\tilde{r}_{t,t+1}}), \end{aligned} & \text{otherwise.} \end{cases} \quad (36)$$

$$T_t^{\widetilde{W}}(\cdot) := \begin{cases} \begin{aligned} &(\widetilde{W}_t(1 - \alpha_t - \vartheta_t) + W_t(\nu_t - \alpha_t - \vartheta_t)) \\ &\times (\delta_t e^{z_{t+1}} + (1 - \delta_t)e^{\tilde{r}_{t,t+1}}) \\ &- \Theta(\widetilde{W}_t(1 - \alpha_t - \vartheta_t) + W_t(\nu_t - \alpha_t - \vartheta_t)), \end{aligned} & \text{if } \widetilde{W}_t(1 - \alpha_t - \vartheta_t) > \\ \delta_t e^{z_{t+1}} + (1 - \delta_t)e^{\tilde{r}_{t,t+1}}, & W_t(\alpha_t + \vartheta_t - \nu_t), \\ 0, & \text{otherwise.} \end{cases} \quad (37)$$

Any annuitisation is reflected by increased consumption in equation (11), hence the input for the utility from consumption becomes $U_C(\alpha_t(W_t + \widetilde{W}_t) + P_t + Y_t, G_t, t)$ as consumption is based on not only wealth drawdown and Age Pension, but now also annuity payments. Annuities need to be handled differently in the means tests. Annuities are assessed based on the income they provide with a deduction for part of the annuity value (Department of Social Services, 2016). The definition of annuity income for the income test is

$$y_t - \frac{a_{t_x}(y_t)}{e_x - t_x}, \quad (38)$$

where t_x is the annuity purchasing time and e_x is the life expectancy at time t_x . The assessment value in the income test is therefore the annuity payments received each year, adjusted for an income test deduction determined at the time of purchase. In the asset test, the value of the annuity is assumed to be equal to the original purchase price of the annuity with a linear yearly value decrease until the life expectancy age is reached, i.e.

$$\max \left(a_{t_x}(y_t) - \frac{a_{t_x}(y_t)}{e_x - t_x}(t - t_x), 0 \right). \quad (39)$$

These rules cause some implications to the model, as it will require additional state variables in terms of annuity purchase price and annuity purchasing time (which complicates

the problem definition further as it is allowed to add on to annuities later in retirement). Even if a numerical solution using Least-Squares Monte Carlo method technically could handle the additional states, it is preferred to avoid this as the additional state variables will have a very minor impact on the value function but are prone to unnecessary regression errors. To avoid this, the calculations in equations (38) and (39) are approximated. The annuity income deduction for the income test is approximated with a constant proportion $\Upsilon = 0.9$ of the annuity payments, which tends to match the deduction amount in the income test very well over time as illustrated in Figure 1. The annuity value in the asset test is approximated using equation (27) to re-value the annuity in the actuarially correct way at the current time given the known annuity payments, thus the asset test annuity assessment approximately equals $a_t(Y_t)$. This approximation is correct at the time of purchase, but overestimates the value of the annuity after that. However, the asset test tends to impose less penalty on the Age Pension received compared with the income test, and only binds for lower levels of wealth (Andréasson and Shevchenko, 2017a). The overestimation of the annuity value in the asset test therefore has a very minor impact on the Age Pension received, and does not have a material effect on the optimal annuitisation.

The means test pension functions now need to be updated. The function for the income test becomes

$$P_I := P_{\max}^d - (P_D(W_t) + Y_t(1 - \Upsilon) - L_I^d) \varpi_I^d, \quad (40)$$

and the function for the asset test is

$$P_A := P_{\max}^d - (W_t + a_t(Y_t) - L_A^{d,h}) \varpi_A^d. \quad (41)$$

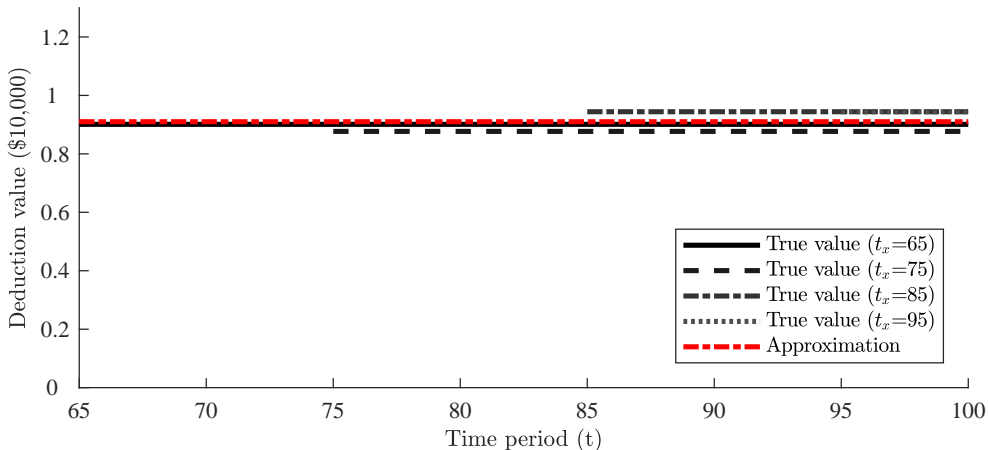


Figure 1: The value of the annuity income deduction in the income test (for annuities purchased at different ages) compared to the approximation of the annuity income deduction by a constant proportion $\Upsilon = 0.9$ of annuity payments. The annual annuity payment is set to \$10,000.

Finally, the extended model requires some additional constraints which are reflected in the admissible action space. The lower bound of consumption drawdown now contains any annuity payments, which the retiree can choose to save in the deposit account instead

of consuming them. The total drawdown (drawdown for consumption and for allocation to annuity purchase) cannot exceed total wealth, although allocation to annuity purchases can exceed total wealth if the retiree decides to save part of the Age Pension and annuity payment. The admissible action space is therefore updated to

$$D_t(x_t) = \left\{ \pi_t(x_t) \in \mathcal{A} \mid \alpha_t \geq \frac{\bar{c}_d - P_t - Y_t}{W_t + \widetilde{W}_t}, \alpha_t + \vartheta_t \leq 1 \right\}. \quad (42)$$

3.2 Extension 2 - Scaling housing and reverse mortgages

The second main extension to the model allows the retiree to either scale the housing by selling the current home and acquiring a new one of a different size or standard. Although downsizing is more common in retirement, especially in the case of a spouse passing away (Olsberg and Winters, 2005; Asher et al., 2017), in our model the retiree is allowed to both up- and downscale at any point in time by making a decision $\tau_t \in [-1, \infty)$ for $t = t_0 + 1, \dots, T - 1$. A positive value represents the proportion of the current house value added to housing (upsizing from the current house), where the transition function for housing becomes

$$T_t^H := H_{t+1} = H_t(1 + \tau_t). \quad (43)$$

The decision variable is therefore bounded below by the current house value, and the upper bound depends on wealth. Decision is made at the start of each period and any house scaling is assumed to be instantaneous (no delay between the decision, the sale of the house and buying a new one). To capture the illiquid nature of housing assets, a proportional transaction cost applies. This will reflect actual costs associated with a sale of the house, as well as avoiding the risk of the optimal decision being a gradual yearly change in the housing asset. The transaction cost only affects the sale of the house, as any transaction cost for a new purchase is assumed to be absorbed by the other party.

The retiree can also choose to take out a reverse mortgage on the house. The assumptions of the loan structure is based on Shao et al. (2015), although not limited to a single payment at issuance. Define $L_t \in \mathcal{L} = \mathbb{R}^+$ to be the loan value at time t . The retiree can at any time make the decision to loan a certain proportion $l_t \in [0, \bar{L}_t]$ of the house value up to the threshold \bar{L}_t . The loan is based on a variable interest rate, where the outstanding loan amount accumulates over time. It is possible to increase an existing loan at any time up to \bar{L}_t , which is given by

$$\bar{L}_t = H_t I(t) \quad (44)$$

where $I(t)$ is a function for the principal limit (maximum LVR ratio) which changes with age, and is defined as

$$I(t) = 0.2 + 0.01(\min(85, t) - 65). \quad (45)$$

The maximum LVR therefore starts at 20% for age 65, which increases with 1% per year to a maximum of 40% at age 85¹⁰. The retiree is not liable to repay part of the loan in the case where the loan value exceeds the LVR or the house value due to accumulated interest (cross-over risk). If the retiree dies, or decides to sell the house, any remaining house value after loan repayments goes to wealth (and can be bequeathed). As Australian reverse

¹⁰The parameterisation follows ‘Equity Unlock Loan for Seniors’ offered by the Commonwealth Bank, but does not impose a minimum or maximum dollar value for the loans.

mortgages include a ‘no negative equity guarantee’¹¹, the retiree (or the beneficiaries) are not required to cover any remaining negative house asset if $L_t > H_t$ at time of death or if the house is sold¹². From the lender’s point of view, this results in two main risks: house price risk and longevity risk. If the house price decreases, or the retiree lives too long so that the loan value accumulates over the house value, the lender is liable for any losses unless these are forwarded to a third party via insurance. Increased interest rates can also speed up compounding of the loan, which increases crossover risk. These risks are in practice covered with a mortgage insurance premium rate added to the loan, in addition to any lending margin required by the lender. The loan-value state therefore requires a transition function, and evolves as

$$T_t^L := L_{t+1} = (L_t \mathbb{I}_{\tau_t=0} + l_t H_t (1 + \tau_t)) e^{\tilde{r}_{t,t+1} + \varphi} \quad (46)$$

where $\mathbb{I}_{\tau_t=0}$ is the indicator symbol if no changes to house assets are made, and φ represents the lending margin and mortgage insurance premium combined. In the case $\tau_t \neq 0$, any outstanding loan value must be repaid, hence the loan is reset and a new loan can be taken out subject to the new house value. The costs of any decision (house transaction cost, the difference in house assets in case of scaling and repayment of loan) is reflected in the wealth process. Let

$$\Delta_t^H = \frac{l_t H_t (1 + \tau_t) - \mathbb{I}_{\tau_t \neq 0} (H_t (\tau_t + \eta) + L_t)}{W_t + \widetilde{W}_t} \quad (47)$$

represent all changes to wealth from house scaling and reverse mortgage decisions as a proportion of current wealth, where $\mathbb{I}_{\tau_t \neq 0}$ is the indicator symbol if any scaling of housing occurs and η is the proportional transaction cost. Then the transition functions for the wealth states can be defined as

$$T_t^W(\cdot) := \begin{cases} \begin{cases} W_t (1 - \alpha_t - \Delta_t^H) \\ \times (\delta_t e^{z_{t+1}} + (1 - \delta_t) e^{\tilde{r}_{t,t+1}}), \end{cases} & \text{if } \widetilde{W}_t (1 - \alpha_t - \Delta_t^H) > W_t (\alpha_t + \Delta_t^H - \nu_t), \\ \begin{cases} (W_t + \widetilde{W}_t) (1 - \alpha_t - \Delta_t^H) \\ \times (\delta_t e^{z_{t+1}} + (1 - \delta_t) e^{\tilde{r}_{t,t+1}}), \end{cases} & \text{otherwise,} \end{cases} \quad (48)$$

$$T_t^{\widetilde{W}}(\cdot) := \begin{cases} \begin{cases} (\widetilde{W}_t (1 - \alpha_t - \Delta_t^H) + W_t (\nu_t - \alpha_t - \Delta_t^H)) \\ \times (\delta_t e^{z_{t+1}} + (1 - \delta_t) e^{\tilde{r}_{t,t+1}}) \end{cases} & \text{if } \widetilde{W}_t (1 - \alpha_t - \Delta_t^H) > \\ - \Theta(\widetilde{W}_t (1 - \alpha_t - \Delta_t^H) + W_t (\nu_t - \alpha_t - \Delta_t^H)), & W_t (\alpha_t + \Delta_t^H - \nu_t), \\ \delta_t e^{z_{t+1}} + (1 - \delta_t) e^{\tilde{r}_{t,t+1}}, & \\ 0, & \text{otherwise.} \end{cases} \quad (49)$$

In addition to new transition functions, the bequest function needs to include the house asset after any reverse mortgage has been repaid, and becomes $U_B(W_t + \widetilde{W}_t, \max(H_t - L_t, 0))$.

¹¹The guarantee is still subject to default clauses which can negate the guarantee, such as not maintaining the property, malicious damage to the property by the owner, failure to pay council rates and failure to inform the provider that another person is living in the house.

¹²Even if the possibility exists, it will not be optimal to sell the house if the net house asset is negative as the retiree will give up ‘free’ housing utility and receive no extra wealth. The exception is a significant upsizing at old age, which is not very likely.

New constraints need to be imposed on the control variables. The option to take out (or add to) a reverse mortgage is bounded from above by the difference of any outstanding mortgage and the LVR, hence

$$l_t \leq \max \left(0, \bar{L}_t - \frac{L_t \mathbb{I}_{\tau=0}}{H_t(1 + \tau_t)} \right). \quad (50)$$

Note that if the control variable for scaling housing is not 0, any outstanding reverse mortgage must be paid back in full and a new reverse mortgage is available against the new house value. The max-condition in the formula is to ensure that the upper bound does not fall below the lower bound to ensure a feasible solution. For the scaling of housing, an upper bound for how much the house asset can be increased is determined by available wealth after costs associated with selling the current house (and repaying any outstanding reverse mortgage) and allocating additional wealth to the new house

$$\tau_t \leq \frac{W_t + \widetilde{W}_t - \mathbb{I}_{\tau \neq 0}(\eta H_t + L_t)}{H_t}. \quad (51)$$

The lower bound is simply -1 , since the retiree cannot downscale further than selling the house and not buying a new one, and the cost associated with the sale is reflected in the transition functions for the state variables. Finally, drawdown still needs to cover any consumption that exceeds the Age Pension received, but no longer has an upper bound of 1 as the maximum amount possible to draw down depends on how housing decisions and new mortgages affect current wealth. This is fully covered in the budget constraint

$$\alpha_t(W_t + \widetilde{W}_t) + \mathbb{I}_{\tau \neq 0}(\eta H_t + L_t) - l_t H_t(1 + \tau_t) - (W_t + \widetilde{W}_t) \leq 0. \quad (52)$$

The constraint specifies the total effect control variables have on wealth, where it ensures that the wealth is enough to cover consumption and housing costs in the case of scaling housing (including repaying any outstanding reverse mortgage) and grows if an additional reverse mortgage is taken out.

As for parameterisation, the transaction cost of selling is set to $\eta = 6\%$ as in Nakajima (2017) and Shao et al. (2015). The markup to the interest rate is set according to Chen et al. (2010), with $\varphi = 0.0242$, but does not require a starting cost to access the loan. In addition, it is assumed there is no current debt on the house (or that it is used as security for other liabilities), and that there are no monthly fees in addition to φ .

4 Results

In this section we present the optimal decisions for each extension. Extension 1 is focused on optimal annuitisation over time in retirement and is based on a single household, since the joint mortality risk of a couple household increases the price of the annuity (thus making it less attractive to annuitise). Extension 2 is focused on changes to house assets in retirement subject to age and total wealth. The numerical solution for both extensions utilises the Least-Squares Monte Carlo algorithm presented in Appendix A implemented in Matlab. On a modern computer (Intel i7, 16GB RAM) using 10,000 sample paths the calculation takes approximately few days, subject to the number of control variables and extension type.

4.1 Extension 1: annuitisation

The optimal annuitisation is expected to differ from previous research due to a number of reasons. In both Iskhakov et al. (2015) and Büttler et al. (2016), the retirement is modelled with a starting wealth that is assumed to be fully consumed and cannot be bequeathed. This means that the level of annuitisation identified given a certain wealth, age and parameters is optimal on a relative basis compared to alternative investment options in order to maximise consumption each time period. Since the model utilised in this paper was calibrated to the behaviour of Australian retirees (Andréasson et al., 2017), where wealth appears in the bequest function, the annuitisation rate is expected to be lower. Similarly, as consumption declines with age, any desired consumption above the consumption floor which can be covered with annuitising early in retirement is not as desirable at older age. In many cases this excess consumption is fully covered by the Age Pension payments. In addition to this, Iskhakov et al. (2015) do not allow for a risk-free rate, hence the annuity is the only (non-reversible) option to access risk-free investments. As the extension model allows the retiree to choose a risk-free allocation, this option can decrease the annuitisation further.

Figure 2 presents the results for the optimal annuitisation at different ages for the cases when risk-free investment is available (default case) and is not available. The latter corresponds to $\delta_t = 1, \forall t$. Each scenario assumes that no prior annuitisation has been done. The case where no risk-free asset is available is almost identical to the default case, indicating that retirees prefer annuities over risk-free investments due to the mortality credit. If the interest rates happen to be higher than normal, then allocation to annuities is slightly higher as well, even if the interest rate is expected to revert back to normal levels. The annuitisation level peaks around age 75 and quickly decreases with age, and is close to constant for higher levels of wealth. Already at age 85 the level of annuitisation is virtually non-existing and stays there. For low wealth levels, where full Age Pension is received, the optimal allocation quickly goes towards zero. A retiree with \$500,000 in liquid wealth at retirement optimally allocates 40% to annuities, which results in approximately \$13,000 in annual annuity payments. If the decision is deferred to age 75, the optimal annuitisation is approximately the same for the same wealth, but the resulting annual payments are higher at \$21,000. Although an Australian retiree has a lower desire for consumption at an older age, the mortality credit at this age is significant and the retiree can access a large boost in yearly consumption for a relatively small wealth sacrifice, resulting in higher overall utility.

To set this in relation to previous research, the results from Iskhakov et al. (2015) suggest on average a higher level of annuitisation, where the range of the authors' different risk preference and return parameters cover the ones calibrated in Andréasson et al. (2017) and used in this paper. The suggested allocation in Iskhakov et al. (2015) is expected to be higher, owing to the constraint that all wealth is to be consumed. That aside, the result confirms the general findings in Iskhakov et al. (2015) and Büttler et al. (2016) – annuitisation is crowded out by the Age Pension and annuitisation increases with wealth but quickly flattens to a constant proportion. Both papers find evidence that the means tests impact annuitisation, especially when binding. This can be seen as the decreasing annuitisation rate around \$200,000 in Figure 2, which represents the transition from full to partial Age Pension. By annuitising at this (or lower) wealth level, no more Age Pension can be received by decreasing assets held, but the annuity payments lead to less Age Pension due to the income test. When a partial pension is received, however, any

annuity payments are only partly assessed in the income test, hence annuitisation is still high until full Age Pension is received. The means-tested Age Pension thus effectively crowds out annuitisation at lower wealth, but not for wealthier households. There are no indications of high sensitivity to means-tested thresholds, however, other than decreasing annuitisation rate when the means tests bind.

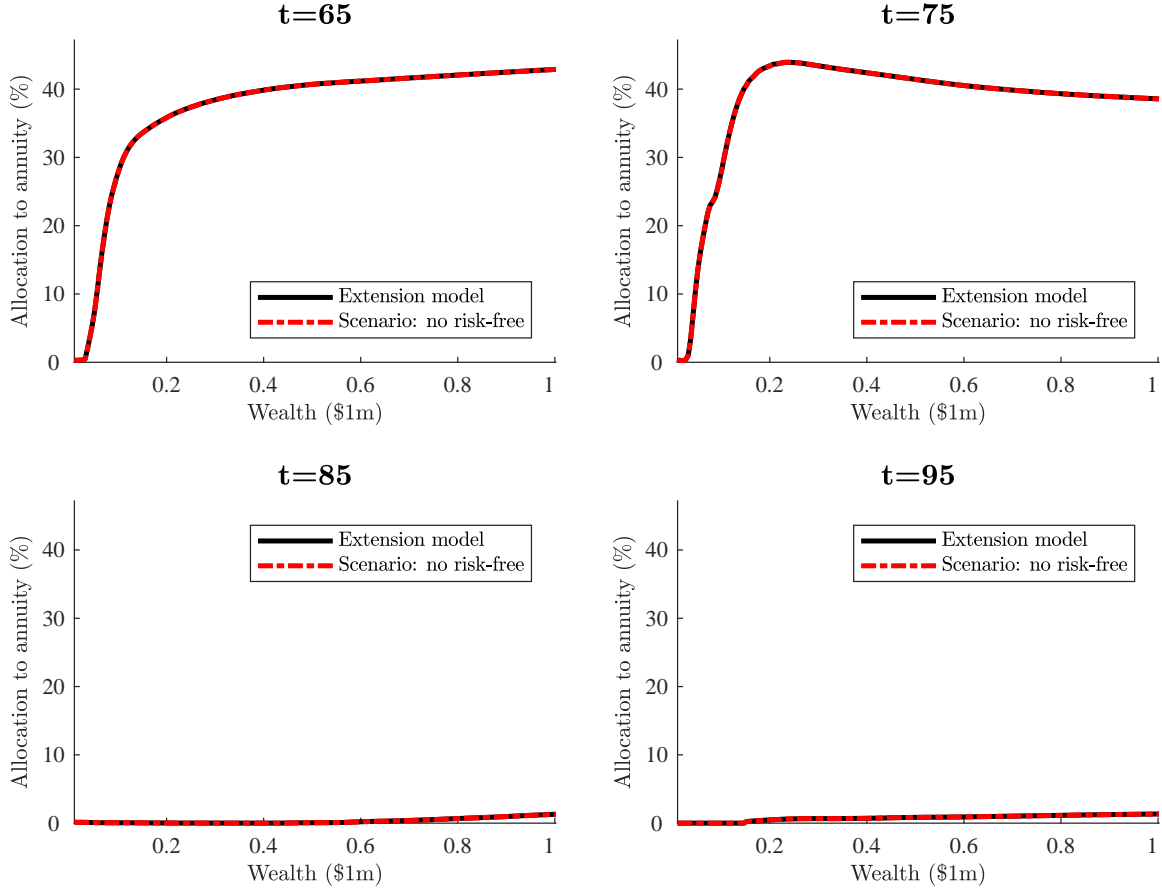


Figure 2: Optimal annuitisation at age t years versus liquid wealth $W_t + \widetilde{W}_t$ and assuming no prior annuitisation. The risky asset and interest rate are assumed to follow the expected paths. In addition to the default case (“extension model”) where the risk-free asset is available, the scenarios when no risk-free asset is available are presented.

Contrary to Iskhakov et al. (2015) and Bütler et al. (2016), but similar to Milevsky and Young (2007), in our model the retiree is allowed to purchase annuities at any time, rather than only at time of retirement ($t = 65$). Figure 3 shows the total annuity allocation for a given initial liquid wealth during the retirement. In order to calculate this, it is assumed the retiree follows the optimal control and that wealth grows with the expected return (the risky asset and interest rate follow the expected paths). This gives a very different perspective of optimal annuitisation than what is seen in Figure 2. Households with lower wealth now have a significant proportion of annuities. This is due to the effect of quickly decreasing consumption with age, hence Age Pension payments accumulate and wealth increases, which is then partly annuitised. It is sub-optimal for poor households to annuitise, but if their wealth grows it is optimal to annuitise at a later stage of retirement. The calculations of total annuitisation in retirement can also be used to evaluate when in retirement annuitisation is optimal. The longer the retiree waits to annuitise, the larger

the mortality credit will be in relation to price (due to the higher death probability), but on the other hand the desired excess consumption decreases towards the consumption floor. By deferring the choice to annuitise, the assets can instead be used to generate investment returns. Figure 4 shows the cumulative wealth allocated to annuities with age. The majority of total annuitisation happens during the first year in retirement, and then increases slightly between ages 70 to 85. This supports the findings in Milevsky and Young (2007) who showed that it is optimal to have immediate partial annuitisation, which also increases with wealth. Early annuitisation indicates that it is not optimal to delay annuities in order to get increased risky exposure. Iskhakov et al. (2015) found that deferred annuities are more attractive to less wealthy retirees owing to the cheaper price. The extension model does not get the same result, due to the lack of additional mortality credit when using immediate annuities, compared to deferred annuities which are purchased before the annuity payments start.

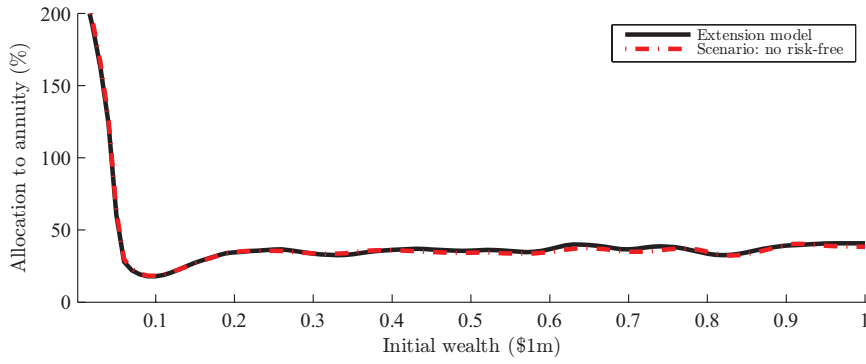


Figure 3: Optimal total dollar amount allocation to annuities over the life time in retirement relative to initial liquid wealth.

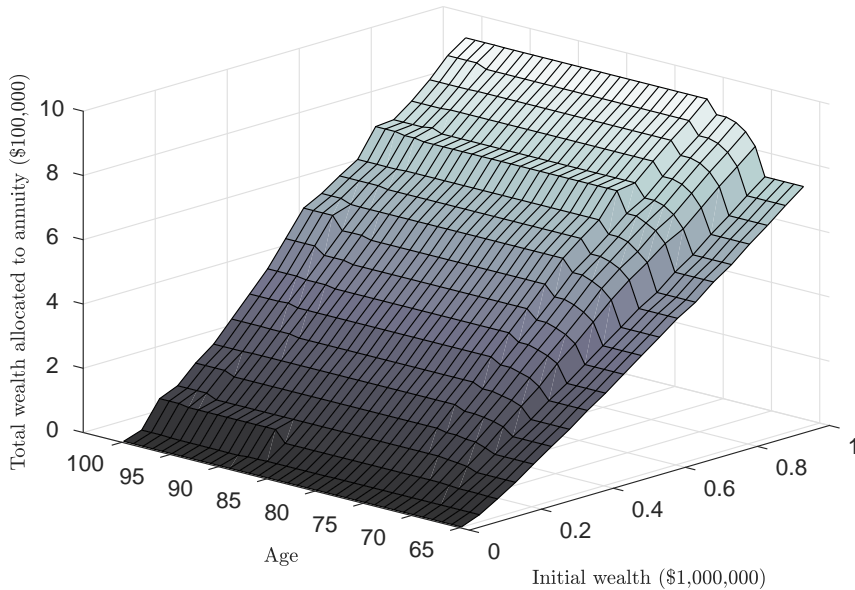


Figure 4: Optimal allocation to annuities over time in retirement given initial liquid wealth $W_{t_0} + \bar{W}_{t_0}$, assuming no previous annuitisation at t_0 .

It should be noted that the optimal annuitisation is an upper bound due to the assumption of no commercial loadings. If a commission or management fee was present, this would make the annuity less desirable. In addition, since wealthier households tend to live longer than less wealthy (De Nardi et al., 2010), the annuitisation is potentially underestimated for the wealthier households and overestimated for the less wealthy household. As the model does not include medical expenses at older age, nor aged care, it can be argued that additional annuitisation is optimal when these costs are included. At the same time, since entering aged care (i.e. a retirement village) attracts rather large one-time costs, this can decrease the optimal level of annuitisation. The finding that annuitisation is optimal only early in retirement might also change in this case.

4.2 Extension 2: scaling housing and reverse mortgage

The purpose of this extension of the model is to evaluate whether scaling housing or accessing home equity is optimal in retirement. In order to test this, it is important that the retiree starts with the optimal house asset at the time of retirement. If not, then the solution might suggest scaling housing just to meet the initial optimal ratio of house assets to liquid wealth. This does not reflect whether it is optimal to scale housing in retirement however, only that it is optimal with a certain level of housing assets in relation to wealth once retired. The retiree therefore starts with the optimal house asset at retirement for a given liquid wealth, and the wealth paths and optimal controls are then simulated until terminal time T . Figure 5 shows the wealth, housing and reverse mortgage paths throughout retirement based on optimal decisions and the expected return on risky assets. Three different levels of total initial wealth at retirement are considered: \$1m, \$2m and \$4m where it is optimal to allocate approximately 80%, 77.5% and 75% respectively into housing for a single household. The single household is chosen as the relative risk aversion for housing is slightly lower than risk aversion for consumption. As can be seen, it is not optimal to downscale housing in any of the cases, while all of them take advantage of the reverse mortgage to keep liquid wealth at a relatively constant level. The loan value is added on to during retirement when required, but also grows based on the interest accumulated.

The optimal reverse mortgage as a proportion of the house value decreases with wealth, and increases with the house value. Irrespective of house value, the loan proportion starts at the same value for households with no wealth. One might expect that the proportion would be less for a higher house value, as this would still access more wealth for the retiree, but this is not the case. However, the higher the house value, the more liquid wealth the retiree can have and still optimally takes out a reverse mortgage. This confirms the results in Chiang and Tsai (2016), who found that as age increases, and the higher the initial wealth and house price are, the more the retiree is willing to use reverse mortgages. Figure 6 shows the optimal loan proportion for different house values in relation to liquid wealth for single households, where the proportion in relation to wealth has a very linear relationship. A less wealthy household, which might need the wealth more than a wealthier household, generally should not take out a reverse mortgage unless the house value is substantially higher than the liquid wealth. Each line in Figure 6 reaches zero before it equals the optimal liquid wealth given the house value, hence a reverse mortgage is never optimal until wealth is drawn down enough to differ significantly from the house asset. The same relationship holds true for couple households, although at a slightly higher wealth level than for singles. When comparing the optimal loan proportion over

time in retirement, the initial maximum level of approximately 10% increases yearly, but flattens out around year 80 and then remains constant at approximately 20%. The LVR threshold therefore never binds when a loan is created, given the calibrated parameters. It is reasonable to expect that if the risk aversion or preferences for bequest decreases, then the optimal loan value might increase. The optimal reverse mortgage in the solution is also an upper bound, as additional commercial loadings such as a fee to initiate the loan might apply in reality. However, a reverse mortgage could theoretically be refinanced if interest rates drop, thus any costs associated with the loan can be lowered that way.

If the retiree's housing asset is significantly less than optimal, then the solution will quickly suggest that the retiree should scale (or acquire) housing assets to get close to the optimal level. However, the opposite does not hold true. If the retiree starts with housing assets significantly larger than optimal, then it is not optimal to downscale, with the exception if the retiree has close to no wealth at all but significant wealth in the house asset. In general, it is therefore never optimal to downscale housing in retirement, not even when reverse mortgages are not available. Only in the case of an event which would incur a significant cost, such as a medical issue, would it be reasonable to downscale. This event is not modelled, however, and would be a result of the budget constraints due to the threshold of the loan value, rather than to maximise utility. It is not optimal to upscale housing once retired either, with the exception of very low house assets (\sim \$100,000 or less) which only reflects the desire to get close to the initial optimal ratio rather than an actual upsizing decision. The reason why downsizing housing is not optimal stems from a combination of the high cost associated with the sale of the house while housing is included in the bequest (hence wealth is given up by downsizing), and that the calibrated consumption floor is already covered by Age Pension payments. If the retiree wants to access just part of his/her house wealth, then downsizing the house will first incur a transaction cost on the full home value, even if the retiree only downscales slightly. To access 10% of the housing wealth, he/she needs to give up 6% of this equity in costs. It is therefore much more economical to take out a reverse mortgage. At the same time, housing utility is received based on the value of the house, even if there is an outstanding reverse mortgage. By utilising the reverse mortgage the retiree can therefore keep a high housing utility, while still accessing the equity. The retiree will give up bequeathable wealth as the loan value accumulates interest, but the funds received from the reverse mortgage can either be invested at a higher (although risky) return and the loss of utility in bequest is partly compensated with higher housing utility through retirement.

A final interesting outcome is that the additional decision variables impact the optimal housing in relation to wealth. With access to the housing asset in retirement, it is now optimal to allocate slightly more towards housing as can be seen in Figure 7. This effect is due to the possibility of 'hiding' away assets in a family home, and then tapping into these assets using the reverse mortgage. It should also be noted that unlike other jurisdictions, Australia does not tax the imputed rent of housing, further adding to the bias towards holding housing as an asset. A retiree can avoid having assets included in the means tests by over allocating to housing assets, and therefore receive additional 'free' wealth from the Age Pension. As the liquid wealth is consumed, it can be replenished by taking out a reverse mortgage, while still accessing the Age Pension.

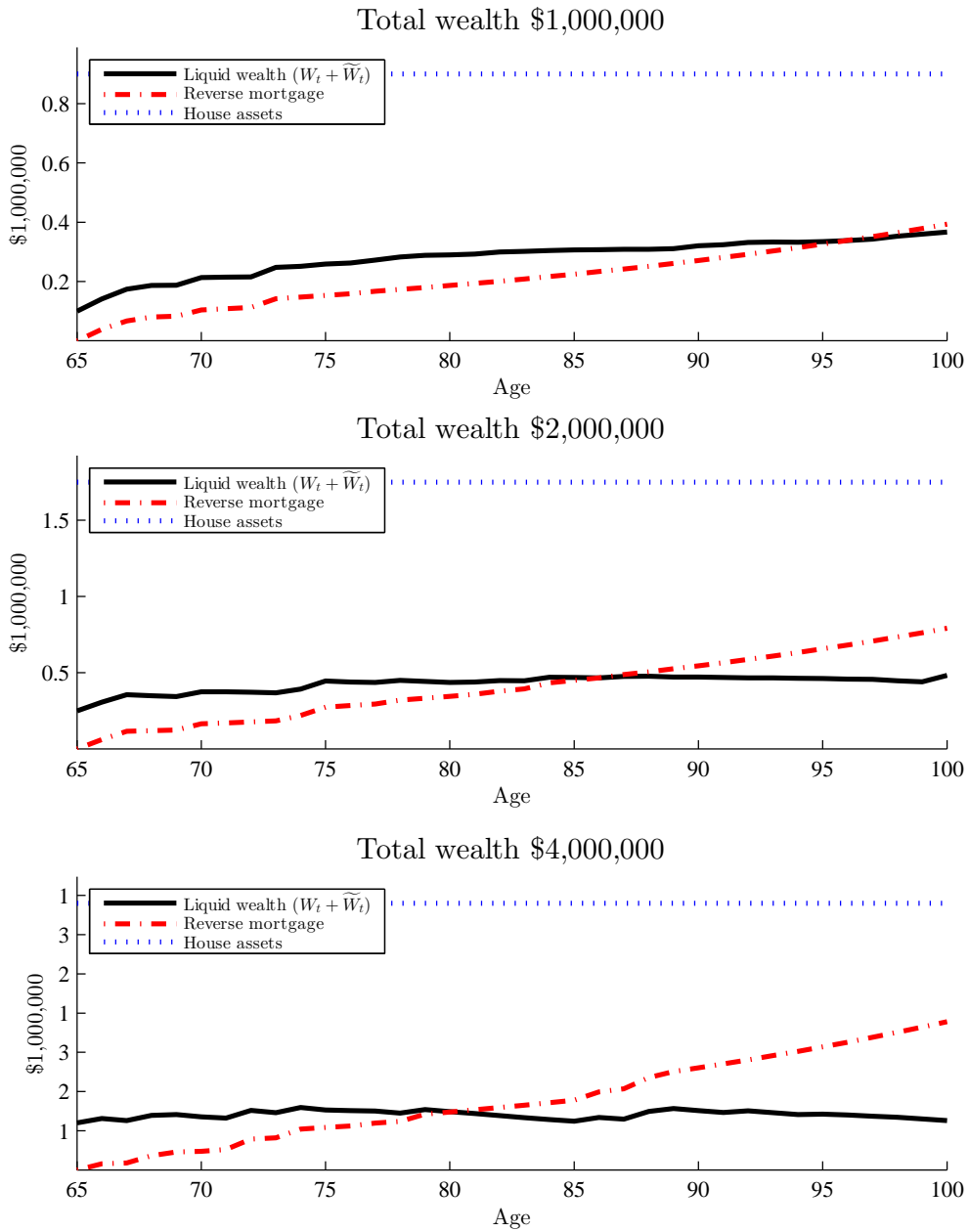


Figure 5: Wealth, house and reverse mortgage paths in retirement given low, medium and high initial total wealth.

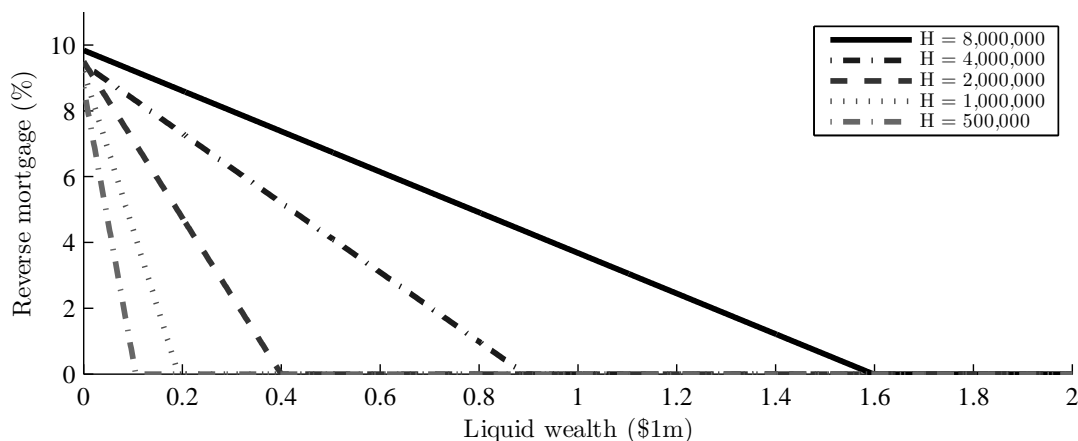


Figure 6: Optimal proportion of reverse mortgage given housing wealth and liquid wealth at retirement age t_0 for a single household.

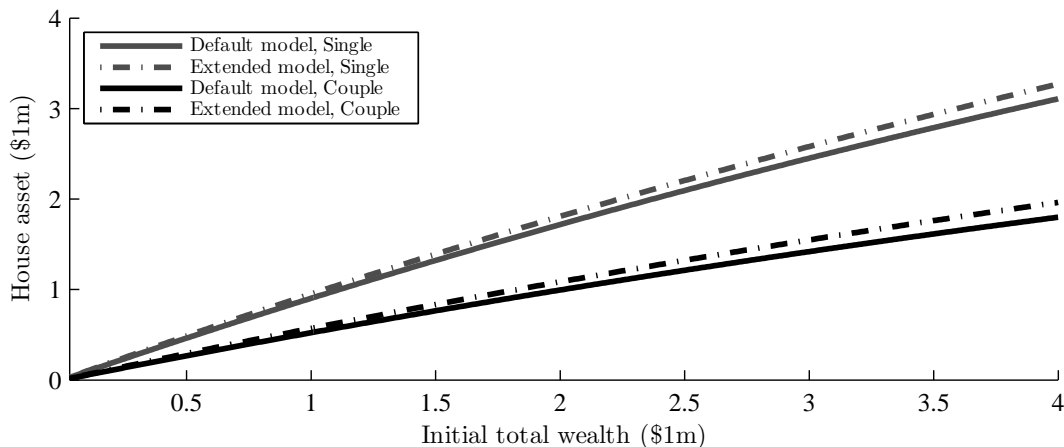


Figure 7: Optimal allocation to housing at retirement for the default case compared to extension model 2 where decisions for scaling housing and reverse mortgage are available.

5 Conclusion

In this paper we developed a retirement model with the option to annuitise wealth, and option to scale housing or access the home equity with a reverse mortgage. It was then evaluated whether such options were optimal during retirement, although not jointly, in relation to the means-tested Age Pension.

In general, the optimal annuitisation in a realistic retirement model setup verifies previous research performed with more restricted models. The means-tested Age Pension crowds out annuitisation, and the alternative to allocating wealth to an annuity is preferred over the risk-free rate. Even when a partial Age Pension is received it is optimal to have partial annuitisation, although the annuitisation decreases quickly when liquid wealth is around the threshold of the full Age Pension. For wealthier households, the annuity payments are much higher than the partial Age Pension received, so even if ‘free’ wealth is given up the retiree is better off annuitising. The total allocation to annuities, or the point in time, is not known at the time of retirement and depends on the realisations

of stochastic factors in retirement.

An annuity provides a significant discount in terms of mortality credits, where the additional utility is higher compared with the alternative to invest the funds in risky assets and annuitise at a later stage in retirement. As consumption decreases with age, this could make annuitisation less desirable, and the results indicate this to be true once the retiree passes age 85 even though the mortality credit is higher at older ages. It is optimal to annuitise sooner rather than later as it is cheaper to store wealth in an annuity rather than risk-free investments. In the Australian setting, it is not optimal to take a one-off decision to annuitise, but rather to gradually increase allocation in the first ten years in retirement, and to annuitise additional wealth depending on the wealth evolution.

A retiree is in general better off utilising a reverse mortgage rather than downsizing the house, despite the accumulated interest of the loan. By keeping a house that is larger than optimal while drawing down the housing assets, the retiree still receives utility from living in the house, while it is still partly bequeathable. The additional utility from this outweighs the cost of an outstanding reverse mortgage. A reverse mortgage does therefore not necessarily benefit a retiree financially, unless the retiree can access additional Age Pension payments by ‘hiding’ assets in the family home, but it does help maximising utility throughout retirement. The optimal decisions are, however, subject to wealth levels and housing assets, where wealthier retirees with more housing assets optimally access a higher proportion reverse mortgage than less wealthier households.

The developed model can be adapted to account for entering aged care facilities and to suit the retirement phase in other countries which is a subject of future research.

Acknowledgment

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A Bias-corrected Least-Squares Monte Carlo

Numerical solution of the model is based on the endogenous state Least-Squares Monte Carlo (LSMC) algorithm studied in detail in Andréasson and Shevchenko (2017b). The approximation of the conditional expectation in a value function is made using ordinary least-squares regression, where the basis functions consist of the fourth order ordinary polynomials of the state and control variables. The exception is for model Extension 2, where the state variable for the outstanding loan value L_t is replaced with the covariate $\max(0, H_t - L_t)$ as this is how it appears in the bequest function. To avoid the transformation bias in the algorithm, the combination of the Smearing Estimate and Controlled Heteroskedasticity is used (for details, see Andréasson and Shevchenko (2017b)). The solution is run with 10,000 sample paths, and the optimisation of the variables is performed with a grid search algorithm. A brief description of the algorithm is provided below.

Let $t = 0, 1, \dots, N$ correspond to equispaced points in the time interval $[0, T]$. Consider the standard discrete dynamic programming problem with the objective to maximise the expected value of utility-based total reward function

$$V_0(x) = \sup_{\pi} \mathbb{E} \left[\beta^N R_N(X_N) + \sum_{t=0}^{N-1} \beta^t R_t(X_t, \pi_t) \mid X_0 = x; \pi \right], \quad (53)$$

where $\pi = (\pi_t)_{t=0,\dots,N}$ is a control, $X = (X_t)_{t=0,\dots,N}$ is a controlled state variable, R_N and R_t are reward functions and β is a time discount factor over a time step. The evolution of the state variable is described by a transition function

$$X_{t+1} = \mathcal{T}_t(X_t, \pi_t, Z_{t+1}), \quad (54)$$

where $Z = (Z_t)_{t=1,\dots,N}$ is a disturbance term with realisation z_t , hence the state of the next period depends on the state of the current period, the control decision and the realisation of the disturbance term. This type of problem can be solved with backward recursion of the Bellman equation

$$\begin{aligned} V_t(x) &= \sup_{\pi_t} \left\{ R_t(x, \pi_t) + \mathbb{E} \left[\beta V_{t+1}(X_{t+1}) \mid X_t = x; \pi_t \right] \right\}, t = N - 1, \dots, 0, \\ V_N(x) &= R_N(x), \end{aligned} \quad (55)$$

but is very computationally intensive when using a quadrature based method for evaluation of expectation in (55) as the number of states, stochastic and control variables increase. The idea behind utilising the LSMC method is to approximate the conditional expectation in Equation (55)

$$\Phi_t(X_t, \pi_t) = \mathbb{E} [\beta V_{t+1}(X_{t+1}) \mid X_t; \pi_t], \quad (56)$$

by a regression scheme with independent variables X_t and randomised π_t , and response variable $\beta V_{t+1}(X_{t+1})$. The approximation of the function is denoted as $\hat{\Phi}_t$. Specifically, a discretised version of the control randomisation technique and LSMC algorithm with realised values from Kharroubi et al. (2014) is used with improvements from Andréasson and Shevchenko (2017b). First, the random state, control and disturbance variables are generated and the evolution of state is calculated for $X_t^m, \pi_t^m, m = 1, \dots, M, t = 0, \dots, T$ (forward simulation) as in Algorithm 1, where *Rand* corresponds to random sampling from some distribution that could be designed for the specific problem. Then, the problem is solved with backward induction as in Algorithm 2. To avoid difficulties in the approximation of the value function due to the extreme curvature of utility functions, a transformation $H(x)$ that has a similar shape as the value function is required (i.e. the Constant Relative Risk Aversion $H(x) = x^\gamma/\gamma, \gamma < 0$). At each point in time $t < T$, the value function is approximated using ordinary least-squares regression

$$\begin{aligned} H^{-1}(\beta V_{t+1}(X_{t+1}^m)) &= \mathbf{\Lambda}'_t \mathbf{L}(X_t^m, \pi_t^m) + \epsilon_t^m, \\ \epsilon_t^m &\stackrel{iid}{\sim} F_t(\cdot), \quad \mathbb{E}[\epsilon_t^m] = 0, \quad \text{var}[\epsilon_t^m] = \sigma_t^2, \quad m = 1, \dots, M, \end{aligned} \quad (57)$$

where $\mathbf{L}(X_t^m, \pi_t^m)$ is a vector of basis functions, $\mathbf{\Lambda}_t$ the regression coefficient vector and H^{-1} the inverse of the transformation function. Thus

$$\Phi_t(X_t, \pi_t) := \int H(\mathbf{\Lambda}'_t \mathbf{L}(X_t, \pi_t) + \epsilon_t) dF_t(\epsilon_t). \quad (58)$$

Here, $F_t(\epsilon_t)$ is the distribution of disturbance term ϵ_t . The corresponding estimated regression coefficient vector is denoted $\hat{\mathbf{\Lambda}}_t$, and the empirical distribution of residuals

$$\hat{\epsilon}_t^m = H^{-1}(\beta V_{t+1}(X_{t+1}^m)) - \hat{\mathbf{\Lambda}}'_t \mathbf{L}(X_t^m, \pi_t^m) \quad (59)$$

can be used to perform this integration. If heteroscedasticity is present in residuals, then conditional variance can be modelled as

$$\text{var}[\epsilon_t \mid X_t, \pi_t] = [\Omega(\mathcal{L}'_t \mathbf{C}(X_t, \pi_t))]^2, \quad (60)$$

where $\Omega(\cdot)$ is some positive function, \mathbf{L}_t is the vector of coefficients and $\mathbf{C}(X_t, \pi_t)$ is a vector of basis functions that can be estimated e.g. as described in Andréasson and Shevchenko (2017b). Then, the estimate of $\Phi_t(X_t, \pi_t)$ is given by

$$\hat{\Phi}_t(X_t, \pi_t) = \frac{1}{M} \sum_{m=1}^M H \left(\hat{\Lambda}'_t \mathbf{L}(X_t, \pi_t) + \Omega(\hat{\mathbf{L}}'_t \mathbf{C}(X_t, \pi_t)) \frac{\hat{\epsilon}_t^m}{\Omega(\hat{\mathbf{L}}'_t \mathbf{C}(X_t^m, \pi_t^m))} \right). \quad (61)$$

The optimal control for each sample can now be calculated, and the value function needs to be updated with the optimal paths for t, \dots, T as the control at time t affect the future states. The algorithm is then iterated for all samples M backward in time to the starting time $t = 0$.

Algorithm 1 Forward simulation

```

1: for  $t = 0$  to  $N - 1$  do
2:   for  $m = 1$  to  $M$  do
      [Simulate random samples ]
3:      $X_t^m := \text{Rand} \in \mathcal{X}$  ▷ State
4:      $\tilde{\pi}_t^m := \text{Rand} \in \mathcal{A}$  ▷ Control
5:      $z_{t+1}^m := \text{Rand} \in \mathcal{Z}$  ▷ Disturbance
      [Compute the state variable after control]
6:      $\tilde{X}_{t+1}^m := \mathcal{T}_t(X_t^m, \tilde{\pi}_t^m, z_{t+1}^m)$  ▷ Evolution of state
7:   end for
8: end for

```

Algorithm 2 Backward solution (Realised value)

```

1: for  $t = N$  to  $0$  do
2:   if  $t = N$  then
3:      $\hat{V}_t(\tilde{\mathbf{X}}_t) := R_N(\tilde{\mathbf{X}}_t)$ 
4:   else if  $t < N$  then
      [Regression of transformed value function]
5:      $\hat{\Lambda}_t := \arg \min_{\Lambda_t} \sum_{m=1}^M \left[ \Lambda'_t \mathbf{L}(X_t^m, \tilde{\pi}_t) - H^{-1}(\beta \hat{V}_{t+1}(\tilde{X}_{t+1}^m)) \right]^2$ 
      Approximate conditional expectation  $\hat{\Phi}_t(X_t, \tilde{\pi}_t)$  using eq. (61)
6:     for  $m = 1$  to  $M$  do
7:        $\hat{X}_t^m := \tilde{X}_t^m$ 
      [Optimal control]
8:        $\pi_t^*(\hat{X}_t^m) := \arg \sup_{\pi_t \in \mathcal{A}} \left\{ R_t(\hat{X}_t^m, \pi_t) + \hat{\Phi}_t(\hat{X}_t^m, \pi_t) \right\}$ 
      [Update value function with optimal paths]
9:        $\hat{V}_t(\hat{X}_t^m) := R_t(\hat{X}_t^m, \pi_t^*(\hat{X}_t^m))$ 
10:       $\hat{X}_{t+1}^m := \mathcal{T}_t(\hat{X}_t^m, \pi_t^*(\hat{X}_t^m), z_t^m)$ 
11:      for  $t_j = t + 1$  to  $N - 1$  do
12:         $\hat{V}_{t_j}(\hat{X}_{t_j}^m) := \hat{V}_t(\hat{X}_t^m) + \beta^{t_j - t} R_{t_j}(\hat{X}_{t_j}^m, \pi_{t_j}^*(\hat{X}_{t_j}^m))$ 
13:         $\hat{X}_{t_j+1}^m := \mathcal{T}_{t_j}(\hat{X}_{t_j}^m, \pi_{t_j}^*(\hat{X}_{t_j}^m), z_{t_j}^m)$ 
14:      end for
15:       $\hat{V}_t(\hat{X}_t^m) := \hat{V}_t(\hat{X}_t^m) + \beta^{N-t} R_N(\hat{X}_N^m)$ 
16:    end for
17:  end if
18: end for

```
