Partial Derivatives

Partial derivatives are the same as normal derivatives, just with additional variables involved that are treated as constants.

Ex: Find all the partial derivatives of:

a) \( g(x, y, z) = (z + e^{xy}, yz) \)

b) \( f(x, x) = \begin{cases} y \sin x & x \neq 0 \\ y & x = 0 \end{cases} \)

Solution: a) let \( g_1 = z + e^{xy} \) \( g_2 = yz \)

then \( \frac{dg_1}{dx} = ye^{xy} \quad \frac{dg_1}{dy} = 2yxe^{xy} \quad \frac{dg_1}{dz} = 1 \)

\( \frac{dg_2}{dx} = 0 \quad \frac{dg_2}{dy} = z \quad \frac{dg_2}{dz} = y \)

are all six partial derivatives.

The matrix given by:

\[
Df = \begin{pmatrix}
\frac{dg_1}{dx} & \frac{dg_1}{dy} & \frac{dg_1}{dz} \\
\frac{dg_2}{dx} & \frac{dg_2}{dy} & \frac{dg_2}{dz}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
ye^{xy} & 2yxe^{xy} & 1 \\
0 & z & y
\end{pmatrix}
\]

is called the total derivative.

b) For this question, we must separate the \( x \neq 0 \) and the \( x = 0 \) cases.

For \( x \neq 0 \): \( \frac{df}{dx} = \frac{y \cos x - y \sin x}{x^2} \) by the quotient rule

For \( x = 0 \): \( \frac{df}{dx} = \lim_{h \to 0} \frac{f(0 + h, y) - f(0, y)}{h} \)

\( = \lim_{h \to 0} \frac{y \sin h - y}{h} \)

\( = \lim_{h \to 0} y \left( \frac{\sin h}{h} - 1 \right) \)

\( = y \lim_{h \to 0} \frac{\sin h - 1}{2h} \) by \( \ell \) 'Hopital's rule.

\( = y \lim_{h \to 0} \frac{\sin h}{h} \) by \( \ell \) 'Hopital's rule again.

\( = y \cdot 0 \)

\( = 0 \)

and \( \frac{df}{dy} = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \)
Chain Rule

\[ \frac{D(g(f))}{Df} = Dg(f) \cdot Df. \]

**Example:** Let \( f(x, y, z) = (x^2y + z, \sin(xy)) \) and \( g(a, b) = (e^a, ab, 1) \). Find \( D(f(g)) \) at \((0, 0)\).

**Solution:**

\[
\begin{pmatrix}
\frac{df}{dx} & \frac{df}{dy} & \frac{df}{dz} \\
\frac{df}{dx} & \frac{df}{dy} & \frac{df}{dz}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
x^2y & x^2 & 1 \\
y \cos(xy) & x \cos(xy) & 0
\end{pmatrix}
\]

Sub in \((x, y, z) = g(0, 0) = (1, 0, 1)\):

\[
Df(g(0, 0)) = \begin{pmatrix}
0 & 1 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

Next use \( g(a, b) = (e^a, ab, 1) \):

\[
\begin{pmatrix}
\frac{dg}{da} & \frac{dg}{db} \\
\frac{dg}{da} & \frac{dg}{db} \\
\frac{dg}{da} & \frac{dg}{db}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
e^a & 0 \\
0 & b \\
0 & 0
\end{pmatrix}
\]

At \((a, b) = (0, 0)\):

\[
Dg(0, 0) = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

\[\circ D(f(g(0, 0))) = Df(g(0, 0)) \cdot Dg(0, 0)\]

\[
= \begin{pmatrix}
0 & 1 & 1 \\
0 & 1 & 0
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 \\
0 & b \\
0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

Note that these chain rule questions are often asked in a way that makes it difficult to tell what your two functions \( f \) and \( g \) are.

You can also be asked to compare your chain rule answer to what you get when you just substitute \( g \) into \( f \) (or vice versa) and differentiate directly. Both methods will always give the same answer.
Chain Rule #2

When you only need 1 derivative in your answer, a quicker version of the chain rule is:

\[
\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} + \frac{dz}{dy} \frac{dy}{dt} \quad \text{+ extra if more variables.}
\]

Ex: Let \( z = e^{x^2} \cdot \tan y \) and \( x = t^2 \) and \( y = \sin(\pi t) \). Find \( \frac{dz}{dt} \) at \( t = 0 \).

Solution: At \( t = 0 \) we have \( x = 0^2 = 0 \) and \( y = \sin(\pi \cdot 0) = 0 \), then,

\[
\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} + \frac{dz}{dy} \frac{dy}{dt} \quad \text{where} \quad \begin{align*}
\frac{dz}{dx} &= ye^{x^2} + 1 \quad \Rightarrow \quad \frac{dz}{dx} \bigg|_{(0,0)} = 1 \\
\frac{dz}{dy} &= xe^{x^2} \quad \Rightarrow \quad \frac{dz}{dy} \bigg|_{(0,0)} = 0 \\
\frac{dx}{dt} &= 2t \quad \Rightarrow \quad \frac{dx}{dt} \bigg|_{(0)} = 0 \\
\frac{dy}{dt} &= \pi \cos(\pi t) \quad \Rightarrow \quad \frac{dy}{dt} \bigg|_{(0)} = -\pi
\end{align*}
\]

A common mistake in the above question is to sub in the wrong point.
You will notice at the start of the question I found the point \((x,y)=(0,0)\)
which I needed, but was not given.

If you are not given a point you still need to make sure your final
answer has no \( x \) or \( y \) variables (or the equivalent in your case).
For example, if \( t = 0 \) was not given \( t = 0 \) in the above question,
then my answer would be:

\[
\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} + \frac{dz}{dy} \frac{dy}{dt} \quad \text{where} \quad \begin{align*}
\frac{dz}{dx} &= ye^{x^2} + 1 \quad \Rightarrow \quad \sin(\pi t) e^{t^2 \sin(\pi t)} + 1 \\
\frac{dz}{dy} &= xe^{x^2} \quad \Rightarrow \quad t^2 e^{t^2 \sin(\pi t)} \\
\frac{dx}{dt} &= 2t \\
\frac{dy}{dt} &= \pi \cos(\pi t)
\end{align*}
\]

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