SECTION A

(60 marks) (Use a separate book for SECTION A)

[5 Marks]  Q1. a) Use appropriate double angle formulas to simplify the following expression:

\[ \frac{4(2\cos^2 x - 1)\sin x \cos x}{\sin 4x}. \]

b) Convert the expression \( f(x) = 5 \cos x + 12 \sin x \) into the form

\[ f(x) = A \sin(x + \phi) \]

for some \( A \geq 0 \) and \( 0 \leq \phi < 2\pi \).

[10 Marks] Q2. a) Calculate the following limits:

(i) \( \lim_{x \to 1} \frac{1}{x^2 - 1} \left( \frac{1}{x + 3} - \frac{2}{3x + 5} \right) \),

(ii) \( \lim_{x \to 0} x \cot(x) \).

b) Find a number \( a \) such that the following function is everywhere continuous:

\[ f(x) = \begin{cases} \cos(ax) & \text{for } x \leq 1, \\ \frac{1}{2} e^{x-1} & \text{for } x > 1. \end{cases} \]

c) (i) Show that the equation \( \tan x + \frac{1}{3}x = 0 \) has at least one solution in the interval \( \frac{3\pi}{4} \leq x \leq \pi \). State clearly what results you are using.

(ii) By considering the derivative or otherwise, show that the equation \( \tan x + \frac{1}{3}x = 0 \) has exactly one solution in the interval \( \frac{3\pi}{4} \leq x \leq \pi \). State clearly what results you are using.
[10 Marks] **Q3.**

a) Use the definition to find the derivative at the point $x = 0$ of the function $f$ determined by $f(x) = \sqrt{x + 1}$.

b) Consider the continuous function given by

$$f(x) = (x - 1)e^{-x}.$$  

(i) Find an equation for the tangent line at the point $x = 1$ for this function; leave the expression in exact form. Is the slope positive or negative?

(ii) Find the absolute maximum and minimum of this function on the closed and bounded interval $[0, 4]$.

c) Consider the relation given by $x^2 + (y - x)^3 = 9$. Use implicit differentiation to find the slope of the tangent at the point $(1, 3)$. That is, find $y'(1)$.

[10 Marks] **Q4.**

a) (i) Evaluate $\int_{0}^{1} \frac{x}{1 + x^2} \, dx$.

(ii) Using partial fractions, find $\int_{-1}^{1} \frac{1}{4 - x^2} \, dx$.

b) (i) Use integration by parts to evaluate $\int_{0}^{\pi} x \cos x \, dx$.

(ii) Find all anti-derivatives of the function $f$ given by $f(x) = x \cos x$; that is, find $\int x \cos x \, dx$.

c) Find the volume generated by revolving the region enclosed by the $x$-axis and the graph of the function $f$ given by $f(x) = x^2 - 1$ about the $x$-axis.
[15 Marks] Q5.  

a) Use Gaussian elimination to solve the system of equations

\[
\begin{align*}
- x_1 + x_2 + 2x_3 &= 8 \\
- x_1 - 2x_2 + 3x_3 &= 1 \\
3x_1 - 7x_2 + 4x_3 &= 10
\end{align*}
\]

b) Solve the system of equations

\[
\begin{align*}
- x_1 - 2x_2 + 3x_3 &= 1 \\
3x_1 - 7x_2 + 4x_3 &= 10
\end{align*}
\]

c) Let the points \( A = (-2, -2, 0) \), \( B = (-2, 0, -2) \) and \( C = (0, -2, -2) \) be the vertices of a triangle in \( \mathbb{R}^3 \).

(i) Write down the vectors \( \mathbf{u} = \overrightarrow{AB} \), \( \mathbf{v} = \overrightarrow{BC} \) and \( \mathbf{w} = \overrightarrow{AC} \).

(ii) Find a vector \( \mathbf{n} \) that is orthogonal to both \( \mathbf{u} \) and \( \mathbf{v} \).

(iii) Find a parametric equation of the plane on which the triangle lies.

(iv) Find a cartesian equation of the plane on which the triangle lies.

(v) Find a parametric equation for the line containing the points \( A \) and \( C \).


a) Consider the differential equation

\[ \frac{dy}{dx} = e^{-x} (y^2 + 1). \]

(i) Determine the general solution.

(ii) Determine the solution that satisfies the initial condition \( y(0) = 1 \).

b) Determine the general solution of the differential equation

\[ \frac{dy}{dx} - 5y = e^x. \]

Extra

1. In how many visually distinct ways can you arrange the letters of RUSSIANREVOLUTION?

2. \( \sum_{k=1}^{n} \frac{n^2}{6} = \frac{1}{6} n(n+1)(2n+1) \). Prove by induction.
PART A

(Use a separate book.)

1. a) Use appropriate double angle formulae to simplify the following expression:

\[
\frac{2 - 2 \cos 2\theta}{4 \sin^2 \theta}.
\]

b) Convert the expression \( f(x) = \cos x + \sqrt{3} \sin x \) into the form

\[
f(x) = A \sin(x + \phi)
\]

for some \( A \geq 0 \) and \( 0 \leq \phi < 2\pi \). In particular, find \( A \) and \( \phi \).

2. a) Calculate the following limits:

\[
\text{[10 Marks]}
\]

(i) \( \lim_{x \to 2} \frac{x + 2}{x^2 - 4} \)

(ii) \( \lim_{x \to 3} \frac{\sqrt{x} - 2}{x - 3} \)

b) Find a number \( a \) such that the following function is everywhere continuous:

\[
f(x) = \begin{cases} 
\sin \frac{x}{|x|}, & \text{for } x < 0, \\
x^2 + a, & \text{for } x \geq 0.
\end{cases}
\]

c) (i) Show that the equation \( e^{-x} = x^3 \) has at least one solution in the interval \( 0 \leq x \leq 1 \).

(ii) By considering derivatives or otherwise, show that the equation \( e^{-x} = x^3 \) has exactly one solution in the interval \( 0 \leq x \leq 1 \). State clearly what results you are using.
3. a) Use the definition of the derivative to find the derivative of the function \( f(x) = x^3 \) at the point \( x = 2 \).

b) At pressure \( P \) atmospheres, a certain fraction \( f \) of a gas decomposes. The quantities \( P \) and \( f \) are related, for some positive constant \( K \), by the equation

\[
\frac{4f^2P}{1 - f^2} = K.
\]

(i) Use implicit differentiation to determine \( \frac{df}{dP} \).
(ii) Show that \( \frac{df}{dP} \) is always negative.
(iii) What does the result in part (ii) mean in practical terms?

c) Consider the function

\[
f(x) = \frac{2x - 4}{x + 1}.
\]

(i) Find the tangent line of the graph of this function at the point \( x = 0 \).
(ii) Find the absolute minimum of this function in the interval \([0, 2]\).

4. Consider the system of linear equations

\[
\begin{align*}
x_1 + 3x_2 + x_3 &= 7, \\
2x_1 + 8x_2 + 3x_3 &= 17, \\
x_1 + 5x_2 + 3x_3 &= 13.
\end{align*}
\] (1)

a) Write the system of equations (1) as an augmented matrix.

b) Use elementary row operations to reduce the matrix from part (a) to reduced row echelon form.

c) Which columns are pivot columns?

d) Determine the basic variables and any free variables and solve the system (1) for \( \mathbf{x} = (x_1, x_2, x_3) \)
5. a) Show that
\[ \frac{1}{2(1 + x^2)} \]
is an anti-derivative of
\[ \frac{x}{(1 + x^2)^2}. \]

b) Use the result from part (a), and the Fundamental Theorem of Calculus, to evaluate the definite integral
\[ \int_0^2 \frac{x}{(1 + x^2)^2} \, dx. \]

c) Use integration by parts, twice, to evaluate
\[ \int_0^\pi t^2 \sin t \, dt. \]

d) A table leg, of height 100 cm, has a circular cross-section whose radius \( r \) at a height \( y \) above the ground is given by
\[ r = 3 + \cos(\pi y/25). \]
Write down an integral for the volume of the table leg. [DO NOT TRY TO EVALUATE THE INTEGRAL].

6. Consider the first order, ordinary differential equation
\[ \frac{dy}{dx} = x + xy. \]  
(2)

a) Using the fact that this equation is separable, determine its general solution.

b) Equation (2) can also be classified as a first order, linear, ordinary differential equation. As such it can be solved using an integrating factor. Show that the integrating factor for equation (2) is
\[ e^{-x^2/2}. \]
Using this result (or otherwise) solve equation (2) subject to the condition that \( y(0) = 1 \).